

A Detailed Examination of Methods for Unifying, Simplifying and Extending Several Results About Self-Justifying Logics

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Abstract

This paper will develop a single framework for unifying, simplifying and extending our prior results about axiom systems that retain a partial knowledge of their own consistency, via an axiomatic declaration of self-consistency. Its perhaps single most surprising new result will be its exploration of a viable alternative to conventional reflection principles.

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1 Introduction

Let α denote an axiom system, and d denote a deduction method. The ordered pair (α, d) will be called **Self Justifying** when:

- i one of α 's theorems states that the deduction method d , applied to the system α , will produce a consistent set of theorems, and
- ii the axiom system α is in fact consistent.

For any (α, d) , it is easy to construct a second axiom system $\alpha^d \supseteq \alpha$ that satisfies Part-i of this definition. For instance, α^d could consist of all of α 's axioms plus the following added sentence, that we call **SelfRef** (α, d) :

- There is no proof (using d 's deduction method) of $0 = 1$ from the *union* of the axiom system α with *this* sentence “SelfRef (α, d) ” (looking at itself).

Kleene [19] discussed how to encode approximate analogs of SelfRef (α, d) 's self-referential statement. Each of Kleene, Rogers and Jeroslow [19, 35, 17] noted α^d may, however, be inconsistent (despite SelfRef (α, d) 's assertion), thus causing it to violate Part-ii of self-justification's definition.

This problem arises in settings more general than Gödel's paradigm, where α was an extension of Peano Arithmetic. There are many settings where the Second Incompleteness Theorem does generalize [1, 2, 3, 8, 9, 10, 11, 14, 15, 16, 21, 24, 25, 29, 30, 32, 33, 34, 35, 37, 39, 44, 46, 49, 52, 54, 56, 58, 62, 66, 63, 67]. Each such result formalizes a paradigm where self-justification is infeasible, due to a diagonalization issue. Most logicians have hesitated to thus employ a SelfRef (α, d) axiom because $\alpha + \text{SelfRef}(\alpha, d)$ is usually inconsistent ¹.

Our research explored special circumstances [61, 64, 65, 66] where it is feasible to construct self-justifying formalisms. These paradigms involved weakening the properties a system can prove about addition and/or multiplication (to avoid the preceding difficulties). To be more precise, let $Add(x, y, z)$ and $Mult(x, y, z)$ denote two 3-way predicates

¹ Typical ordered pairs (α, d) will have the property that the broader axiom system $\alpha^d = \alpha + \text{SelfRef}(\alpha, d)$ will be inconsistent, even when α is consistent. This is because a standard Gödel-like self-referencing construction will typically produce a proof of $0 = 1$ from α^d , irregardless of whether or not α is consistent.

indicating x , y and z satisfy $x + y = z$ and $x * y = z$. A logic will be said to **recognize** successor, addition and multiplication as **Total Functions** iff it includes 1-3 as axioms.

$$\forall x \exists z \quad Add(x, 1, z) \tag{1}$$

$$\forall x \forall y \exists z \quad Add(x, y, z) \tag{2}$$

$$\forall x \forall y \exists z \quad Mult(x, y, z) \tag{3}$$

We will say a logic system α is **Type-M** iff it contains each of (1) – (3) as axioms, **Type-A** iff it contains only (1) and (2) as axioms, and **Type-S** iff it contains only (1) as an axiom. A system is called **Type-NS** iff it *does not* contain any of these axioms.

Our investigations [61]–[67] began by observing some Type-A systems can recognize their consistency under semantic tableaux deduction, and several Type-NS systems can recognize their Hilbert consistency. Many of these systems were capable of proving Peano Arithmetic’s Π_1 theorems in a language that represents addition and multiplication as the 3-way predicates of $Add(x, y, z)$ and $Mult(x, y, z)$.

Our self-justifying evasions of the Incompleteness Theorem are difficult to further extend primarily because the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris [26, 33, 44, 58] showed natural Type-S systems cannot recognize their own Hilbert consistency. Also, Willard [62, 67, 68] strengthened earlier results of Adamowicz-Zbierski [1, 3] to establish that natural Type-M system cannot recognize their semantic tableaux consistency.

A related class of evasions of the Second Incompleteness Theorem was discovered in [68]. Let us say α is a **Type-Almost-M** axiom system iff α can prove statements (4) and (5) as theorems while treating *none of sentences (1) – (5) as axioms*. (Many axiom systems, that use function symbols “ + ” and “ * ” for formalizing addition and multiplication, fall technically into the Type-Almost-M category.)

$$\forall x \forall y \exists z \quad x + y = z \tag{4}$$

$$\forall x \forall y \exists z \quad x * y = z \tag{5}$$

The preceding is of interest because some surprisingly strong (albeit unusual) Type-Almost-M systems [68] have an ability to verify their Herbrand but not also semantic tableaux consistency.

The proofs in our prior papers were challenging primarily because they required one to separate the local combinatorial methods employed in [59, 64, 66, 68]’s particular applications from the common principles that underlied behind all these works. Our Theorems 5.9, 5.11 and 6.6 will rectify this problem by identifying common components that unite these four paradigms. (Theorems 6.3, 6.10, 6.12, E.1, G.2 and G.3 will then carry on in further directions.)

All these theorems will contain severe limits on their generality, so that the Second Incompleteness Theorem does not contradict them. It is clearly perplexing to imagine how humans are able to motivate themselves to cogitate, without their thought processes possessing some type of *at least tentative* presumption of their own consistency. Our research has thus consisted of an approximately equal effort in exploring both [62, 63, 66, 67]’s new generalizations of the Second Incompleteness Theorem and [59, 61, 64, 65, 66, 68]’s unusual boundary-case exceptions to it.

It is clear every boundary-case exception to the Second Incompleteness Theorem has limited scope because the Incompleteness Theorem is a broadly encompassing result. This paper will, thus, be addressing a challenging near-paradoxical question about the maximal nature of self-justification that can never be resolved in a fully satisfying manner. The Second Incompleteness Theorem is clearly sufficiently central to logic for it to be desirable to know what *partial roads of success* a self-justifying axiom system can obtain.

2 Literature Survey

Two 5-page surveys of the prior literature about the Second Incompleteness Theorem were provided in our articles [64, 66]. This section will present a more abbreviated survey, focusing on only those developments that are particularly germane to the current article.

The study of incompleteness began with four classic papers by Gödel, Löb, Rosser and Tarski [14, 25, 36, 50] and with the Hilbert-Bernays exploration of their derivability conditions [15, 16, 18]. Generalizations of these results for weak axiom systems, such as Q, began with the work of Tarski-Mostowski-Robinson [51] and Bezhboruah-Shepherdson [8].

Some more notation is needed to describe more recent developments. Let x' denote the “successor” operation that maps x onto $x + 1$. A formula $\varphi(x)$ is called [15] a **Definable Cut** for an axiom system α iff α can prove:

$$\varphi(0) \text{ AND } \forall x \{ \varphi(x) \Rightarrow \varphi(x') \} \text{ AND } \forall x \forall y < x \{ \varphi(x) \Rightarrow \varphi(y) \} \quad (6)$$

Definable cuts and their cousins have been studied by an extensive literature [1, 3, 6, 9, 10, 15, 18, 21, 22, 26, 27, 29, 30, 31, 32, 33, 34, 39, 45, 46, 52, 54, 55, 56, 57, 58]. (They are unrelated to Gentzen’s notion of sequent calculus deductive “cut rule”, which uses the word “cut” in a different context).

A Definable Cut $\varphi(x)$ is called **Non-Trivial** relative to an axiom system α iff α cannot prove $\forall x \varphi(x)$, *although it can prove (6)*. Every axiom system α , strictly weaker than Peano Arithmetic, will contain some non-trivial Definable Cut. This cut will have the property that α can verify $\varphi(n)$ for each fixed integer n , although it cannot prove $\forall x \varphi(x)$.

Let $\lceil \Psi \rceil$ denote Ψ 's Gödel number, and $\text{Prf}_\alpha^d(t, p)$ denote that p is a proof of the theorem t from the axiom system α using d 's deduction method. An axiom system α will then be said to recognize its own **Cut-Localized d-Consistency** relative to a Definable Cut φ iff α can prove:

$$\forall p \quad \{ \varphi(p) \Rightarrow \neg \text{Prf}_\alpha^d(\lceil 0 = 1 \rceil, p) \} \quad (7)$$

The recent literature has sought to identify which triples (φ, d, α) have this property. A crucial negative result about Cut-Localized d-Consistency, discovered by Pudlák [33], established a significant generalization of Gödel's Second Incompleteness Theorem. It showed that an axiom system α must be unable to prove (7)'s statement about any of its definable cuts φ , when d represents Hilbert deduction and α is any consistent extension of Q .

Solovay [44] noted how Pudlák's result could be combined with the techniques of Nelson and Wilkie-Paris [26, 58] to establish the following theorem that will often be cited in this paper:

Theorem 2.1 (*Solovay's 1994 Generalization [44] of a 1985 theorem of Pudlák [33] using some of Nelson and Wilkie-Paris [26, 58]'s methods*): Let α denote any axiom system which contains Equation (1)'s Type-S statement and which assures the successor operation always satisfies $x' \neq 0$ and $x' = y' \Leftrightarrow x = y$. Then α will be unable to recognize its own Hilbert consistency, whenever it treats addition and multiplication as 3-way relations satisfying their usual identity, associative, commutative and distributive properties.

Solovay never published any precise proof of Theorem 2.1's hybridizing of the work of Pudlák, Nelson and Wilkie-Paris [33, 26, 58], which he privately communicated [44] to us. A reader can find generalizations of the Second Incompleteness Theorem that are closely related to Theorem 2.1 in papers by Pudlák, Buss-Ignjatovic, Švejdar and Willard [10, 33, 46, 63], as well as in Appendix A of [61].

Other interesting observations that preceded our research were that Wilkie-Paris [58] demonstrated that $\text{I}\Sigma_0 + \text{Exp}$ cannot prove the Hilbert consistency of even the axiom system Q , and that Adamowicz-Zbierski [1, 3] showed that $\text{I}\Sigma_0 + \Omega_1$ satisfied the Herbrandized version of the Second Incompleteness Theorem. Both these results helped stimulate [62, 67]'s semantic tableaux generalizations of the Second Incompleteness Theorem for $\text{I}\Sigma_0$.

A fascinating observation by L. A. Kołodziejczyk [20, 21], about the difference in lengths between semantic tableaux and Herbrandized proofs, also motivated our investigation [68] into some surprising properties of unorthodox encodings for $\text{I}\Sigma_0$. Kołodziejczyk observed [20, 21] that various generalizations of the Second Incompleteness Theorem for $\text{I}\Sigma_0$ and $\text{I}\Sigma_0 + \Omega_i$ in [1, 3, 37, 62, 67, 68] imply that the proof of the Herbrandized version of the Second Incompleteness Theorem can be more complicated than its semantic

tableaux counterpart. This is because there can be an exponential difference between semantic tableaux and Herbrandized proof lengths under extremal circumstances. It was due to Kołodziejczyk’s insightful communications ² that [68] developed an axiom system that was a boundary-case exception to the Herbrandized but not also the semantic tableaux version of the Second Incompleteness Theorem.

The literature on Definable Cuts has centered its evasions of the Second Incompleteness Theorem around Equation (7)’s localization formalism (rather than employing analogs of Section 1’s $\text{SelfRef}(\alpha, d)$ axiom, as we did in [59, 61, 64, 65, 66, 68]). Pudlák [33] proved that essentially every axiom system α of finite cardinality can be associated with a definable cut φ such that α can prove sentence (7)’s validity for φ when d is either the semantic tableaux or Herbrand-styled deductive method.

Pudlák’s theorem is related to Friedman’s observation [13] that for many finite theories S and T , the theory S has an interpretation in T if and only if $\text{IS}_0 + \text{Exp}$ can prove that T ’s Herbrand consistency implies S ’s Herbrand consistency. Several generalizations of these results by Krajíček, Pudlák, Smoryński, Švejdar and Visser appear in [22, 33, 34, 39, 46, 54, 55, 56]. Visser’s article [56] contains an excellent review of this literature, as well as many additional new results. Also, we will see how some of the reflection machines of Beklemishev, Kreisel-Takeuti and Verbrugge-Visser [5, 6, 7, 24, 52, 56] nicely complement Theorem 6.12’s reflection mechanisms in alternate types of intended applications.

It was established by Hájek, Švejdar and Vopěnka [45, 57] that GB Set Theory can construct a definable cut φ where it can prove the statement (7) is valid when d denotes Hilbert deduction and α is ZF Set Theory. This result was surprising because Pudlák [33] showed GB can never verify its own Hilbert consistency localized on a definable cut. (Thus, GB will view its Hilbert consistency as equivalent to ZF’s Hilbert consistency in a global sense *but not* in a cut-localized respect.).

In some sense, Kreisel and Takeuti [24, 47] can be viewed as the first authors to develop a logic recognizing its own consistency using a variant of Equation (6)’s formula. Their results for typed logics formalized a second-order generalization of Gentzen’s sequent calculus that can verify its own consistency, when no sequent calculus deductive cuts are performed. A key aspect of their formalism can be seen as using an analog of (7)’s sentence in an implicit manner. It thus begins by using a set of objects, which we shall call I , that includes all the standard integers plus some allowed *non-standard*

² The Herbrandized and semantic tableaux definitions of an axiom system α ’s consistency are different from each other because the former requires skolemizing α ’s axioms, while the latter permits [12] an existential quantifier elimination rule to replace Skolemization. This distinction can create a potential exponential difference between the lengths of Herbrand and Semantic Tableaux proofs. This insightful observation, due to private communications from L. A. Kołodziejczyk [20], was used in [68] to create an axiom system that satisfied the semantic tableaux but not also Herbrandized version of the Second Incompleteness Theorem.

integers (that can permissibly represent contradictory proofs). Their second-order logic then uses Dedekind’s definition of the natural numbers to construct a subset of I , called “ N ”, which includes all the standard integers and which is disallowed to contain any contradiction proof. We will not go into the details here, but this transition from I to N (with an accompanying relativization of the provability predicate onto N ’s more restricted domain) can be viewed as Kreisel-Takeuti’s analog of (7)’s local consistency statement for [24, 47]’s “CFA” second-order logic.

It is difficult to compare our research (which has relied upon an analog of $\text{SelfRef}(\alpha, d)$ ’s Kleene-like “*I am consistent*” axiom) with the preceding literature that has used various forms of Localized d-Consistency statements. This is because every effort to evade the Second Incompleteness Theorem employs some built-in weakness to evade Gödel’s classic paradigm.

Our work in [59, 61, 64, 65, 68] represented less than a full-scale evasion of the Second Incompleteness Theorem mostly because it was incompatible with treating *as formal axioms* the statements in Equations (3) and (5) that multiplication is a total function³. Some reasons why it was helpful for [59, 61, 64, 65, 68] to employ analogs of Section 1’s SelfRef axiom are that:

- A An axiom’s self-referential “*I am consistent*” declaration allows a formalism to recognize its consistency in a global sense, rather than in the φ –localized sense used by sentence (7) and the analogous Kreisel-Takeuti relativization of their second-order proof predicate.
- B If a logic is employing a deductive method d that lacks a modus ponens rule, as occurs in nearly all self-justifying systems, then it is preferable for it to view its “*I am consistent*” statement as an axiom rather than as a theorem. (This is because weak deductive methods are capable of drawing logical inferences only from axioms when modus ponens is absent.)
- C Analogs of Section 1’s $\text{SelfRef}(\alpha, d)$ ’s “*I am consistent*” axiom have been shown by [59, 61, 64, 65, 66, 68] to at least partially formalize the notion of a logic possessing *an almost* instinctive form of faith in its own internal consistency. (This paper will make it apparent that such an instinctive faith is less than a full-scale proof. Yet, Theorem 6.12 and Remarks 6.13, 6.14 and 6.16 will make it apparent that such formalizations of instinctive faith are also useful.)

³This caveat applies also to our article [68], although its Herbrandized form of self-justification differs from our other papers by retaining a capacity to treat (5)’s statement about multiplication’s totality as a derived theorem *that is not an* axiom. The key point is that theorems are weaker than axioms under Herbrand deduction because only axioms are used as intermediate steps during proofs. This explains intuitively how [68]’s formalism was able to recognize its Herbrandized consistency, while treating (5)’s statement about the totality of multiplication *as a theorem*. (We will return to this subject in Appendix D.)

We emphasize that both virtues and drawbacks of $\text{SelfRef}(\alpha, d)$'s "*I am consistent*" axiom statements have been cited in this paragraph because every effort to evade the Second Incompleteness Theorem can obtain no more than limited levels of success.

The scope of the challenge we face becomes apparent when one realizes $\alpha + \text{SelfRef}(\alpha, d)$ is inconsistent for most (α, d) . This is because $\alpha + \text{SelfRef}(\alpha, d)$ typically satisfies Part-i *but not also* Part-ii of Section 1's definition of a "self-justifying" logic. (Thus, a diagonalization paradigm will typically imply $\alpha + \text{SelfRef}(\alpha, d)$ is inconsistent, as a consequence of it containing $\text{SelfRef}(\alpha, d)$ as an axiom.) This is the reason Kleene, Rogers and Jeroslow [19, 35, 17] were hesitant about the utility of $\text{SelfRef}(\alpha, d)$'s mirror-like axiom sentence. Our goal in [59]-[68] has been to develop generalizations and boundary-case exceptions for the Second Incompleteness Theorem, so as determine exactly which paradigms can support, for example, Theorem 6.12's limited notion of self-justification.

The reason one would anticipate some *limited* exceptions to the Second Incompleteness Theorem to exist is it is hard to imagine how humans can motivate themselves to cogitate without using some variant of self-justification.

3 Generic Configurations

The phrase **Bounded Quantifier** will refer to expressions of the form " $\exists v \leq T$ " or " $\forall v \leq T$ " where T is a term. A formula is called **Fully-Bounded** when all its quantifiers are so bounded. Lemma 3.6 will soon explain how Definition 3.1's formalism can encode conventional arithmetic:

Definition 3.1 Let ξ denote some non-integer indexing superscript (whose properties will be discussed later by Definition 3.4). Then the symbol Δ_0^ξ will denote some fixed special set of fully-bounded formulae that is closed under negation, in a language that will be later called L^ξ . (Thus, if some formula Ψ is a member of Δ_0^ξ then so is $\neg \Psi$.) Items 1-3 formalize how Π_n^ξ and Σ_n^ξ formulae are built in a straightforward manner out of these Δ_0^ξ sub-components:

1. Every Δ_0^ξ formula is considered to be also a Π_0^ξ and Σ_0^ξ formula.
2. For $n \geq 1$, a formula will be called Π_n^ξ iff it can be written in the canonical form of $\forall v_1 \forall v_2 \dots \forall v_k \Phi(v_1, v_2, \dots, v_k)$, where Φ is Σ_{n-1}^ξ .
3. Likewise for $n \geq 1$, a formula will be called Σ_n^ξ iff it can be written in the form of $\exists v_1 \exists v_2 \dots \exists v_k \Phi(v_1, v_2, \dots, v_k)$, where Φ is Π_{n-1}^ξ .

Notation Convention: Our rules for defining ξ , specified later in this section, will never have this superscript designate an integer quantity. This is because integer superscripts have a special meaning under a typed-based hierarchy, not intended here.

Example 3.2 Let L denote a conventional arithmetic language that uses function symbols for denoting addition and multiplication. Below are two examples of Δ_0 -like formulae that invoke Definition 3.1's notation:

- a The symbol “ Δ_0^A ” will denote any fully bounded formula that uses the addition, multiplication and maximum function symbols in an arbitrary manner. (Thus, Δ_0^A corresponds to what many textbooks [15, 18, 23] simply call a “ Δ_0 ” formula.)
- b The symbol “ Δ_0^R ” will denote a class of formulae in L 's language whose bounded quantifiers are allowed to use only the Maximum function symbol. Their bodies, however, may contain any combination of addition, multiplication and maximum function symbols.

Formulae (8) and (9) illustrate the distinction between the Δ_0^A and Δ_0^R classes. Thus, (8) satisfies the first but not second condition (on account of the presence of the multiplication symbol used by its bounded quantifiers). In contrast, (9) is an example of a Δ_0^R formula.

$$\exists y \leq x * x \quad \forall z \leq y * y \quad \exists w \leq y * z \quad : \quad \{ \quad x * y = z + w \quad \} \quad (8)$$

$$\exists y \leq x \quad \forall z \leq y \quad \exists w \leq \text{Max}(y, z) \quad : \quad \{ \quad x * y = z + w \quad \} \quad (9)$$

The distinction between Δ_0^A arithmetic formulae and the unconventional Δ_0^R class may first convey the impression that these two classes have fundamentally different natures. Actually, Lemma 3.6 will show that their relationship is more subtle. This is because its formalism will map Δ_0^A formulae onto Δ_0^R expressions that are equivalent to it under Definition 3.3's Standard-M model — in a context where only the length of these formulae is allowed to possibly grow. This equivalence enabled [68] to construct a natural axiomatic formalism that could recognize its own Herbrandized consistency *but which nevertheless satisfied* the idealized form ⁴ of the semantic tableaux version of the Second Incompleteness Theorem.

Definition 3.3 “Standard-M” *will denote the standard model of integers.*

The reason for our interest in Standard-M is that many pairs of formulae are equivalent under the Standard-M model, while weak axiom systems often cannot formally prove they are equivalent. For instance, this will occur when Example 3.5 examines Definition 3.4's properties.

⁴An axiom system α is defined to satisfy the “**idealized form**” of the semantic tableaux version Second Incompleteness Theorem when no $\beta \supseteq \alpha$ can prove a semantic tableaux proof of $0=1$ from itself is incapable of existing. We will summarize [68]'s formalism and the distinction between Herbrandized and semantic tableaux deduction at the of Appendix D.

Definition 3.4 *A Generic Configuration, often identified by the superscript symbol of ξ , is defined to be a 5-tuple $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ where:*

1. L^ξ is a language that includes logical symbols for “0”, “1”, “2”, “=”, “ \leq ” and for the operation of “Maximum(x,y)”. L^ξ also includes a sufficient number of function and constant symbols so that every integer k can be encoded by some term T_k specifying k ’s value.
2. Δ_0^ξ corresponds to any variation of Definition 3.1’s class of “fully-bounded” formulae that is rich enough to assure that there exists two Δ_0^ξ formulae, henceforth called “Add(x, y, z)” and “Mult(x, y, z)”, for formalizing the graphs of addition and multiplication. (It will generate ξ ’s set of Π_n^ξ and Σ_n^ξ sentences, using Definition 3.1’s 3-part formalism.)
3. B^ξ denotes a “**Base Axiom System**”, whose axiom-sentences are true under the Standard-M model and which is Σ_1^ξ complete. (Thus, B^ξ can prove every true Σ_1^ξ sentence, and it can likewise refute all false Π_1^ξ sentences.)
4. d denotes ξ ’s method of deduction. It is required to be sufficiently conventional to satisfy the usual indirect-implication property ⁵ associated with Gödel’s Completeness Theorem.
5. g denotes a method for encoding the Gödel numbers of proofs.

Example 3.5 Let us recall Example 3.2 defined “ Δ_0^A ” as essentially the conventional textbook notion [15, 23] of an arithmetic “ Δ_0 ” formula. This example will outline how well-known techniques can map every Δ_0^A formula onto a semantically equivalent Δ_0^ξ formula under the Standard-M model.

Our discussion will have Seq(x) denote a function that maps non-negative integers onto binary strings in lexicographic order. Thus Seq(x) maps 0 onto the empty string, the integers 1 and 2 onto the strings of “0” and “1”, the integers 3–6 onto “00”, “01”, “10”, “11”, etc. (Formally, Seq(x) is an operation that maps integer x onto the bit-string that occurs to the immediate right of the leftmost “1” bit in the binary encoding of $x + 1$.)

Given any k -tuple (x_1, x_2, \dots, x_k) , let STRING(x_1, x_2, \dots, x_k) denote the concatenation of Seq(x_1), ... Seq(x_k). For any integers v and w satisfying $v \leq w^2$, it is clear that there exists (x_1, x_2, x_3) where STRING(x_1, x_2, x_3) represents v ’s binary encoding and each $x_i \leq \text{Max}(w, 4)$.

⁵ This is that irregardless of whether or not d contains a built-in modus ponens rule, it does support some form of a (possibly quite lengthy) proof of a theorem Z , when it is able to prove X, Y and $(X \wedge Y) \rightarrow Z$ as theorems.

An example will now illustrate the approximate structure of an inductive methodology for mapping Δ_0^A formulae onto their equivalent Δ_0^ξ counterparts in the Standard-M model. Let $\text{SQUARE}(x_1, x_2, x_3, w)$ be a Δ_0^ξ formula which specifies that $\text{STRING}(x_1, x_2, x_3)$ represents an integer $\leq w^2$. Also, let $\phi^*(x_1, x_2, x_3)$ and $\phi(v)$ represent a pair of Δ_0^ξ and Δ_0^A formulae that are equivalent under the Standard-M model when $\text{STRING}(x_1, x_2, x_3)$ is an encoding for v . Then one possible method for mapping Δ_0^A formulae onto their equivalent Δ_0^ξ counterparts (in the Standard-M model) could map the formula (10) onto (11)'s alternate form:

$$\forall v \leq w^2 \quad \phi(v) \quad (10)$$

$$\begin{aligned} \forall x_1 \leq \text{Max}(w, 2) \quad \forall x_2 \leq \text{Max}(w, 2) \quad \forall x_3 \leq \text{Max}(w, 2) \\ \{ \text{SQUARE}(x_1, x_2, x_3, w) \Rightarrow \phi^*(x_1, x_2, x_3) \} \end{aligned} \quad (11)$$

Lemma 3.6 indicates Example 3.5's translational methodology generalizes easily to all combinations of Δ_0^A inputs and generic configurations ξ , via an approximate inductive generalization of the transition from sentence (10) to (11). (Its procedure essentially performs iteratively a finite number of such transitions, so as to translate all the clauses of an initial Δ_0^A formulae into their Δ_0^ξ counterparts via an inductive methodology. The intuition behind these transitions is they will repeatedly replace a single variable, such as v in sentence (10), with a multiplicity of variables, such as (x_1, x_2, x_3) in (11).)

Lemma 3.6 (Paris-Dimitracopoulos [28]) *For every generic configuration ξ , each Δ_0^A formula can be translated into an equivalent Δ_0^ξ formula in the Standard-M model via a generalization of Example 3.5's process.* (This clearly implies Π_j^A formulae can also be translated into Π_j^ξ expressions.)

Paris-Dimitracopoulos [28] sketched an analog of Lemma 3.6's translation algorithm, using only slightly different notation, that is applicable to any formalism that satisfies Parts (1) and (2) of Definition 3.4. Their formalism thus uses an inductively-iterated analog of the prior example's replacement of a single variable v in formula (10) with (11)'s multiplicity of variables (x_1, x_2, x_3) , so as to perform Lemma 3.6's translation task. It will be unnecessary for a reader to consider the details behind [28]'s Theorem 1 or Lemma 3.6's similar translation mechanism because the remainder of this article will never use them again. Instead, their sole purpose has been to provide an *implicit backdrop* for our results by illustrating how the study of the Π_1^ξ sentences of Definition 3.4's generic configurations provides information about Π_1^A sentences (after the needed translating is done).

Four examples of self-justifying systems that employ Definition 3.4's Π_1^ξ sentences will be illustrated in Appendix D. These examples are too complicated to be examined before Sections 3 – 6 are read. However, the next example should convey some useful intuitions:

Example 3.7 Let x_0, x_1, x_2, \dots and y_0, y_1, y_2, \dots denote sequences defined by:

$$x_0 = 2 = y_0 \quad (12)$$

$$x_i = x_{i-1} + x_{i-1} \quad (13)$$

$$y_i = y_{i-1} * y_{i-1} \quad (14)$$

For $i > 0$, let ϕ_i and ψ_i denote the sentences in (13) and (14) respectively. Also, let ϕ_0 and ψ_0 denote (12)'s sentence. Then $\phi_0, \phi_1, \dots \phi_n$ imply $x_n = 2^{n+1}$, and $\psi_0, \psi_1, \dots \psi_n$ imply $y_n = 2^{2^n}$. Thus, the latter sequence grows at a faster rate than the former. Much of our research has used the difference between the growth rates of x_0, x_1, x_2, \dots and y_0, y_1, y_2, \dots as a motivating example explaining why Equation (2)'s Type-A axiom systems can support a stronger form of boundary-case exception to the semantic tableaux version of the Second Incompleteness theorem than can Type-M systems.

Let $\text{Log}(y_n) = 2^n$ and $\text{Log}(x_n) = n + 1$ thus designate the lengths of the binary codings for y_n and x_n . Then y_n 's coding has a length 2^n , which is *much larger* than the $n + 1$ steps that $\psi_0, \psi_1, \dots \psi_n$ use to define its existence. However, x_n 's length has a smaller size of $n + 1$. These observations are useful because every proof of the Incompleteness Theorem involves a Gödel number z coding a sentence that has a capacity to self-reference its own definition. The faster growing series $y_0, y_1, \dots y_n$ should be intuitively anticipated to have this self-referencing capacity because y_n 's binary encoding has a 2^{n+1} length that dwarfs the size of the $O(n)$ steps used to define its value. Leaving aside [62, 67]'s many details, this fast growth explains roughly why many Type-M logics satisfy the semantic tableaux version of the Second Incompleteness Theorem.

This paradigm also illustrates intuitively why some Type-A systems, employing [59, 61, 64]'s semantic tableaux formalism, can represent boundary-case exceptions to the Second Incompleteness Theorem. This is because such formalisms lack access to Equation (3)'s axiom that multiplication is a total function. (They are unable, thus, to easily construct numbers z that can self-reference their own definitions because they have access only to the slower growing addition primitive.) In particular assuming only that each sentence in the axiom-sequence $\phi_0, \phi_1, \dots \phi_n$ (from Equation (13)) requires a mere two bits for its encoding, the length $n + 1$ of x_n 's binary encoding will be smaller than the length of its defining sequence.

This short length for x_n had motivated [59, 61, 64, 65]'s evasion of the semantic tableaux version of the Second Incompleteness Theorem. It suggested that the self-

referencing needed in a Gödel-like diagonalization argument would stop being feasible when Equation (13)'s slow-growing x_1, x_2, x_3, \dots sequence represents the fastest growth that is possible.

One of the several goals in this article will be to formalize a generalizations of [59, 61, 64, 65]'s self-justifying methodologies by using Definition 3.4's generic configurations. The proofs of our main theorems will, of course, be more subtle than the hand-waving intuitions appearing in this example. For instance, the combined work of Pudlák, Solovay, Nelson and Wilkie-Paris [26, 33, 44, 58] (summarized by Theorem 2.1) raised the subtle issue that no Type-S system can prove a theorem affirming its own Hilbert consistency. Another complication is that the Equation (14)'s implication for proofs that use the multiplication operative has different side effects for Herbrandized and semantic tableaux deduction (on account of Kołodziejczyk [20, 21]'s previously mentioned observations about the potential exponential difference between the lengths of these proofs under extremal circumstances).

Our main theorems will show that self-justifying systems, using four deduction methods, are capable of proving all of Peano Arithmetic's Π_1^ξ theorems. Interestingly, self-justification will be compatible with [64]'s modification of semantic tableaux deduction, that includes a modus ponens rule for Π_1^ξ and Σ_1^ξ type sentences. However, [63] has shown an analogous modus ponens rule for Π_2^ξ and Σ_2^ξ sentences is incompatible with self justification. (Thus, the contrast between our main results and the Second Incompleteness Theorem's generalizations will be quite tight.)

4 Five Helpful Definitions and An Informative Lemma

This section will introduce five definitions and prove a Lemma 4.6 about self justification. This lemma will be weaker than Sections 5 and 6's main results. Its main purpose will be to provide a useful starting example.

Definition 4.1 The symbol “ $E(n)$ ” will denote some term in Definition 3.4's language L^ξ that represents the value 2^n . In using this symbol, we do not presume that L^ξ possesses a function symbol for the exponent operation. Thus if L^ξ has only a function symbol for multiplication, then $E(n)$ could designate the term of “ $2 * 2 * \dots * 2$ ” with n repetitions of “2”. (Alternatively, $E(n)$ can be defined via applying 2^n iterations of the successor function to zero, or by having a special constant symbol designating 2^n 's value. Essentially, any reasonable method can be used to define $E(n)$'s value)

Definition 4.2 Let Υ denote a prenex normal sentence. Then $\text{Scope}_E(\Upsilon, N)$ will denote a sentence identical to Υ except that every unbounded universal quantifier “ $\forall v$ ” is changed to “ $\forall v < E(N)$ ”, and every unbounded existential quantifier “ $\exists v$ ” is changed to “ $\exists v < E(N)$ ”. (No change is made among the bounded quantifiers within the Δ_0^ξ part of the sentence Υ .) For example, if Υ denotes the Π_1^ξ sentence of $\forall v_1 \forall v_2 \dots \forall v_k \phi(v_1, v_2, \dots, v_k)$ then (15) illustrates $\text{Scope}_E(\Upsilon, N)$ ’s form. Likewise if Υ is the Σ_1^ξ sentence of $\exists v_1 \exists v_2 \dots \exists v_k \phi(v_1, v_2, \dots, v_k)$ then (16) illustrates $\text{Scope}_E(\Upsilon, N)$ ’s form.

$$\forall v_1 < E(N) \forall v_2 < E(N) \dots \forall v_k < E(N) \quad : \quad \phi(v_1, v_2, \dots, v_k) \quad (15)$$

$$\exists v_1 < E(N) \exists v_2 < E(N) \dots \exists v_k < E(N) \quad : \quad \phi(v_1, v_2, \dots, v_k) \quad (16)$$

Special Note about Definition 4.2’s Meaning. If Υ is a Δ_0^ξ sentence then $\text{Scope}_E(\Upsilon, N)$ will be equivalent to Υ for every $N \geq 0$ by definition. (This is because Δ_0^ξ formulae contain no unbounded quantifiers that undergo change when Υ is mapped onto $\text{Scope}_E(\Upsilon, N)$.)

More About this Notation: The potentially lengthy syntactic object of “ $\text{Scope}_E(\Upsilon, N)$ ” *will actually not* be used in our physical encodings of proofs. Instead, these encodings will use the more desirably compressed object of “ Υ ” (which has no possibly bulky $E(N)$ term). The *sole function* of $\text{Scope}_E(\Upsilon, N)$ will be for us to speculate about what Boolean value Υ *would theoretically assume* (under the Standard-M model) if Υ ’s quantifiers were modified so that their ranges were changed to be bounded by $E(N)$. (It turns out that $\text{Scope}_E(\Upsilon, N)$ ’s finitized quantifier-range will help simplify our analysis.)

Definition 4.3 A Π_1^ξ or Σ_1^ξ sentence Υ will be called **Good(N)** when the entity $\text{Scope}_E(\Upsilon, N)$ is true under the Standard-M model⁶. Also, a set of Π_1^ξ sentences, denoted as θ , is called **Good(N)** iff all of its sentences are **Good(N)**.

Definition 4.4 If Υ is a Π_1^ξ sentence then $\sharp(\Upsilon)$ will denote the largest integer J such that Υ satisfies the **Good(J)** condition. (It will equal ∞ if Υ satisfies **Good(J)** for all J .) Also, if θ is a set of Π_1^ξ sentences, then $\sharp(\theta)$ will denote the largest J where each sentence in θ is **Good(J)**.

A Very Helpful Start : Several more definitions will be needed before Section 6 can present our strongest results. The remainder of this section will illustrate how the current formalism is already sufficient for introducing a useful starting lemma.

⁶ A quite unusual aspect of Definition 4.3 is that its **Good(N)** condition has opposite properties when it is applied to Π_1^ξ and Σ_1^ξ sentences in one particular respect. This is because for each N , the **Good(N)** condition is weaker than the **Good(∞)** condition for Π_1^ξ sentences, while it is stronger than it for Σ_1^ξ sentences. (For instance, $\forall x \phi(x)$ is stronger than $\forall x < E(N) \phi(x)$, but $\exists x \phi(x)$ is weaker than $\exists x < E(N) \phi(x)$.)

Definition 4.5 Let $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ again denote a generic configuration called ξ , and let us presume that its base axiom system B^ξ is comprised exclusively of Π_1^ξ sentences. Also, let $\beta \supset B^\xi$ denote a second axiom system, comprised also of Π_1^ξ sentences, that (unlike B^ξ) can possibly be inconsistent. (If β is inconsistent then let q_β denote the shortest proof of $0 = 1$ from β .) Then the generic configuration ξ will be called **Tight** if iff *every inconsistent* set of Π_1^ξ sentences $\beta \supset B^\xi$ satisfies the following constraint:

$$\text{Log}(q_\beta) \geq \#(\beta) + 2 \quad (17)$$

Lemma 4.6 will prove $B^\xi + \text{SelfRef}(B^\xi, d)$ satisfies Section 1's self-justification criteria whenever ξ is tight. This "tightness" will clearly fail to be satisfied by many generic configurations. This is because the Second Incompleteness Theorem is a widely encompassing result, which imposes severe restrictions on its allowed exceptions. Lemma 4.6's mini-result will be of interest primarily because it will be generalized substantially in Sections 5 and 6.

Lemma 4.6 *If a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ is tight then $B^\xi + \text{SelfRef}(B^\xi, d)$ will be a consistent self-justifying axiom system.*

Proof Sketch: Our justification of Lemma 4.6's mini-result will be simpler than the next section's proof of Theorem 5.9's stronger result. The current proof will also be kept brief and informal because the same topic will be visited more rigorously during Section 5's discourse.

Let Ψ denote Section 1's $\text{SelfRef}(B^\xi, d)$ sentence. We will omit formalizing Ψ 's exact Π_1^ξ encoding here because Appendix A will provide a more general fixed-point construction, using Definition 5.7's stronger paradigm. The current proof will simply presume Ψ 's fixed point statement can receive a Π_1^ξ encoding under sentence (18), where " $\text{Prf}_{B^\xi + \text{SelfRef}(B^\xi, d)}^d(\lceil 0 = 1 \rceil, p)$ " is a Δ_0^ξ formula, indicating that p is a proof of $0=1$ from $B^\xi + \text{SelfRef}(B^\xi, d)$ under d 's deduction method.

$$\forall p \quad \neg \text{Prf}_{B^\xi + \text{SelfRef}(B^\xi, d)}^d(\lceil 0 = 1 \rceil, p) \quad (18)$$

The Definition 4.4's symbol $\#$ will be helpful at this juncture. The application of ξ 's tightness to (18)'s Π_1^ξ styled encoding will imply ⁷ that (19) must be true when Ψ is false under the Standard-M model and when the shortest proof of $0 = 1$ from $B^\xi + \text{SelfRef}(B^\xi, d)$ is denoted as q .

$$\text{Log}(q) = \#(\Psi) + 1 \quad (19)$$

⁷Equation (19) is easy to justify when one presumes there is an available Π_1^ξ encoding of (18)'s statement, which we call Ψ . (This Π_1^ξ presumption is reasonable because an analog of Ψ 's exact Π_1^ξ encoding will be discussed later by Definition 5.7 and in Appendix A.) The invalidity of Ψ will thus assure the existence of a proof of $0 = 1$ from $B^\xi + \text{SelfRef}(B^\xi, d)$. Moreover, Definition 4.4's notation implies $\#(\Psi)$ will equal $\text{Log}(q) - 1$ when q denotes the shortest proof of $0 = 1$ from $B^\xi + \text{SelfRef}(B^\xi, d)$. The latter shows (19) is valid.

Also Definition 4.4 trivially implies $\#(B^\xi + \Psi) = \#(\Psi)$ (because all of B^ξ 's axioms are true under the Standard-M model). Thus, (19) yields (20).

$$\text{Log}(q) = \#(B^\xi + \Psi) + 1 \quad (20)$$

But the point is that the Tightness constraint's Equation (17), used in the context where β is the axiom system of $B^\xi + \Psi$, implies $\text{Log}(q) \geq \#(B^\xi + \Psi) + 2$. This directly contradicts Equation (20)'s equality, *whenever* the proof “ q ” of $0 = 1$ cited in our discussion *does formally exist*. This contradiction is precisely what is needed to corroborate Lemma 4.6's claim that the axiom system $B^\xi + \text{SelfRef}(B^\xi, d)$ must be consistent. (Thus, $B^\xi + \text{SelfRef}(B^\xi, d)$ must be consistent because otherwise a proof q of $0 = 1$ would exist and have its $\text{Log}(q) \geq \#(B^\xi + \Psi) + 2$ inequality contradict Equation (20).) \square

How Lemma 4.6 May Be Interpreted : We remind the reader that many (but not all) generic configurations will fail to satisfy Lemma 4.6's tightness hypothesis. This is because any configuration satisfying this hypothesis represents one of those unusual boundary-case exceptions to the Second Incompleteness Theorem that are feasible.

Lemma 4.6 was intended to capture the simplest variant of a self-justifying phenomena (that employs Definition 4.4's machinery). Its proof was kept informal because more sophisticated self-justifying formalisms will be explored in Sections 5 and 6. They will apply to four different types of generic configurations, each of whose base axiom systems B^ξ can be made capable of proving all Peano Arithmetic's Π_1^ξ theorems — in a context where these systems use a broader variant of “*I am consistent*” axiom-statement than does Lemma 4.6's “ $\text{SelfRef}(B^\xi, d)$ ” sentence.

5 The First Two Meta-Theorems about Self-Justification

The core theorems in this section will employ the following notation:

1. The symbol θ will denote any recursively enumerable (r.e.) set of Π_1^ξ sentences, henceforth called a **R-View**. (An R-View does not need to be valid under the Standard-M model. It only needs to be r.e.)
2. **RE-Class**(ξ) shall denote the set of all possible “R-Views” θ that can be built out of ξ 's language L^ξ . (This permits both valid and invalid R-Views to appear in RE-Class(ξ). We choose this unrestricted definition because no recursive decision procedure can identify all the true Π_1^ξ sentences in the Standard-M model.)

Definition 5.1 Let ξ denote the 5-tuple $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ representing one of Definition 3.4's generic configurations, and let $\text{RE-Class}(\xi)$ and its R-Views “ θ ” be defined as in the previous paragraph. Then ξ is called **A-Stable** iff each $\theta \in \text{RE-Class}(\xi)$ satisfies the following invariant:

- * If Υ is a Π_1^ξ theorem of axiom system $\theta \cup B^\xi$ via a proof p whose length satisfies $\text{Log}(p) \leq \#(\theta) + 1$ then Υ will satisfy $\text{Good}\{ \frac{1}{2} \#(\theta) \}$.

Remark 5.2 The invariant * states short proofs (with lengths $\leq \#(\theta) + 1$) will produce at least *partially useful* deductions, in that their Π_1^ξ theorems will always satisfy $\text{Good}\{ \frac{1}{2} \#(\theta) \}$, *irregardless of whether or not θ 's axioms are technically true*. (This makes the study of A-stability very interesting unto itself, apart from its applications in the current article).

Theorem 5.11 will show the presence of A-stability, alone, is sufficient for constructing self-justifying systems. This will imply every A-stable configuration must contain some embedded weakness (as every evasion of the Second Incompleteness Theorem always does). On the other hand, Appendix F will explain how A-stability and its “EA-stable” cousin (defined later) are both epistemologically interesting. Thus, A-stability has redeeming features.

Definition 5.3 A Generic Configuration ξ will be called **E-Stable** iff all of the $\theta \in \text{RE-Class}(\xi)$ satisfy **. (This construct is the counterpart for Σ_1^ξ sentences of the Item * in Definition 5.1.)

- ** If Υ is a Σ_1^ξ theorem derived from the axiom system $\theta \cup B^\xi$ via a proof p whose length satisfies $\text{Log}(p) \leq \#(\theta) + 1$ then Υ will automatically satisfy $\text{Good}\{ \frac{1}{2} \lfloor \text{Log}(p) \rfloor - 1 \}$. (This invariant *further implies*⁸ that Υ will also satisfy the $\text{Good}\{ \frac{1}{2} \#(\theta) \}$ criteria.)

Remark 5.4 The invariants * and ** are partially analogous to each other because both imply that if p is a proof short enough to satisfy $\text{Log}(p) \leq \#(\theta) + 1$ then their resulting theorem will satisfy $\text{Good}\{ \frac{1}{2} \#(\theta) \}$. However, there is a distinction between Definitions 5.1 and 5.3, as well. This is because the prior section's Footnote 6 observed that Σ_1^ξ sentences are stronger when they meet a $\text{Good}(N)$ rather than a $\text{Good}(\infty)$ threshold, *while the reverse is true for Π_1^ξ sentences*. Thus **'s short proofs p (satisfying $\text{Log}(p) \leq \#(\theta) + 1$) will have the special property that their theorems Υ will satisfy a “ $\text{Good}\{ \frac{1}{2} \lfloor \text{Log}(p) \rfloor - 1 \}$ ” constraint that is actually stronger than their formal Σ_1^ξ statements.

⁸This point is easy to confirm when one remembers that Σ_1^ξ sentences Υ have the property that $A < B$ implies $\text{Scope}_E(\Upsilon, A)$ is stronger than $\text{Scope}_E(\Upsilon, B)$. It is then obvious that the $\text{Good}\{ \frac{1}{2} \lfloor \text{Log}(p) \rfloor - 1 \}$ criteria implies the validity of $\text{Good}\{ \frac{1}{2} \#(\theta) \}$ for Σ_1^ξ sentences because the invariant ** presumes its Σ_1^ξ theorems have proofs p satisfying “ $\text{Log}(p) \leq \#(\theta) + 1$ ”.

The Appendix D will provide four examples of generic configurations that are either E-stable or A-stable (or often both). Its most prominent example will be a configuration ξ^* that uses semantic tableaux deduction and recognizes addition as a total function. Theorem D.4 will imply such systems can be made self-justifying and able to prove Peano Arithmetic's Π_1^* theorems.

Three more definitions are needed to help introduce our first theorem

Definition 5.5 A generic configuration ξ will be called **EA-stable** iff it is both E-stable and A-stable. (It will thus satisfy both \ast and $\ast\ast$.)

Our next definition is related to the fact that many definitions of consistency are logically equivalent from the perspective of strong enough logics, but they are *often not provably equivalent* from the perspectives of weak logics.

Definition 5.6 Let ξ denote the generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, and α be an axiom system satisfying $\alpha \supseteq B^\xi$. Then α is called **Level(k^ξ) Consistent** when there exists no proofs from α via d 's deduction method of both a Π_k^ξ sentence and of the Σ_k^ξ sentence that is its negation

Most of this article will focus on self-justifying systems that recognize their Level(k^ξ) consistency when k equals 0 or 1. Our next definition will be applied mostly to these two cases.

Definition 5.7 Given any $k \geq 0$, a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ and an axiom system $\beta \supset B^\xi$, the symbol $\text{SelfCons}^k(\beta, d)$ will denote a self-referencing Π_1^ξ sentence declaring $\beta + \text{SelfCons}^k(\beta, d)$'s formal Level(k^ξ) consistency, as is illustrated below by the statement \vdash . (An encoding for $\text{SelfCons}^k(\beta, d)$ will be provided by Appendix A.)

\vdash There exists no two proofs (using deduction method d) of *both* some Π_k^ξ sentence and of the Σ_k^ξ sentence, that represents its negation, from the union of the axiom system β with *this* added sentence “ $\text{SelfCons}^k(\beta, d)$ ” (*looking at itself*).

Remark 5.8 We will focus on Definition 5.7's $\text{SelfCons}^k(\beta, d)$ axiom mostly in the settings where $k = 0$ or 1. This is because $\text{SelfCons}^k(\beta, d)$ will typically be too strong for it to generate boundary-case exceptions to the Second Incompleteness Theorem when $k \geq 2$. It turns out that even when $k = 1$, Definition 5.7's $\text{SelfCons}^k(\beta, d)$ statement will be significantly stronger than the axiomatic declaration \bullet used by Section 1's $\text{SelfRef}(\beta, d)$ axiom. This is because $\text{SelfCons}^1(\beta, d)$ asserts non-existence of simultaneous proofs for a Π_1^ξ sentence and its negation, while $\text{SelfRef}(\beta, d)$ establishes merely the non-existence of a proof of $0 = 1$.

Theorem 5.9 *Let ξ denote a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ that is EA-stable. Then the corresponding axiom system of $B^\xi + \text{SelfCons}^1(B^\xi, d)$ must satisfy Section 1's definition of self-justification.*

Proof. The justification of Theorem 5.9 will be a more elaborate version of Lemma 4.6's mini-proof. It will replace Definition 4.5's Tightness constraint with an EA-stability requirement. It will also replace $\text{SelfRef}(\beta, d)$'s "I am consistent" axiom with a *stronger* $\text{SelfCons}^1(\beta, d)$ statement.

Our proof will focus on showing $B^\xi + \text{SelfCons}^1(B^\xi, d)$ is consistent (and thus satisfies the subtle Part-ii component of Section 1's definition of "Self-Justification"). It will be awkward during our discussion to write repeatedly the expression " $B^\xi + \text{SelfCons}^1(B^\xi, d)$ ". Therefore, " S " will be the abbreviated name for this axiom system. We will also employ the following notation:

1. $\text{Prf}_S(t, p)$ will denote that p is a proof of the theorem t from the above mentioned formalism " S " (using ξ 's deduction method of d).
2. $\text{Neg}^1(x, y)$ will denote that x is the Gödel encoding of a Π_1^ξ sentence and that y is a Σ_1^ξ sentence which represents x 's formal negation.

Appendix A explains how to combine the theory of LinH functions [15, 23, 69] with [61]'s fixed point methods to provide $\text{Prf}_S(t, p)$ and $\text{Neg}^1(x, y)$ with Δ_0^ξ encodings. (A reader can omit examining Appendix A, if he just accepts this fact.) Thus, (21) can be viewed⁹ in this context as being $\text{SelfCons}^1(B^\xi, d)$'s formalized Π_1^ξ statement, declaring the $\text{Level}(1^\xi)$ consistency of S :

$$\forall x \forall y \forall p \forall q \quad \neg \{ \text{Neg}^1(x, y) \wedge \text{Prf}_S(x, p) \wedge \text{Prf}_S(y, q) \} \quad (21)$$

Let Φ denote (21)'s sentence. We will use it to prove Theorem 5.9's claim that S is consistent. Our proof will be a proof by contradiction. It will begin with the assumption that S is inconsistent. This implies Φ is false under the Standard-M model. Hence, Definition 4.4 implies

$$\sharp(\Phi) < \infty \quad (22)$$

Equation (22) thus indicates there exists $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ satisfying (23) (because such a tuple will be a counter-example to (21)'s assertion). The particular (p, q, x, y) satisfying (23)

⁹Our notation convention has the abbreviated formula of " $\text{Prf}_S(t, p)$ " in Equation (21) corresponding to Appendix A's " $\text{SubstPrf}_\beta^d(\bar{n}, t, p)$ " formula, in a context where \bar{n} specifies the Gödel number of the expression $\Gamma^k(g)$ in Appendix A's Equation (36) and where the superscript k within this formula " $\Gamma^k(g)$ " is set equal to 1. This implies that sentence (21) does assert the $\text{Level}(1^\xi)$ consistency of S .

with minimum value for $\text{Log}\{ \text{Max}[p, q, x, y] \}$ can then be easily shown ¹⁰ to also satisfy Equation (24).

$$\text{Neg}^1(\bar{x}, \bar{y}) \wedge \text{Prf}_S(\bar{x}, \bar{p}) \wedge \text{Prf}_S(\bar{y}, \bar{q}) \quad (23)$$

$$\text{Log}\{ \text{Max}[\bar{p}, \bar{q}, \bar{x}, \bar{y}] \} = \sharp(\Phi) + 1 \quad (24)$$

We will now use (23) and (24) to bring our proof-by-contradiction to its conclusion. Let Υ denote the Π_1^ξ sentence specified by \bar{x} . Then $\neg\Upsilon$ will correspond to the Σ_1^ξ sentence denoted by \bar{y} . Also, (23) and (24) imply that both Υ and $\neg\Upsilon$ have proofs such that the logarithms of their Gödel numbers are bounded by $\sharp(\Phi) + 1$. These facts imply that Υ and $\neg\Upsilon$ *both* satisfy $\text{Good}\{ \frac{1}{2} \sharp(\Phi) \}$ under our formalism. (This is because if we take θ in Definitions 5.1 and 5.3 to be simply Φ 's 1-sentence statement, then the invariants of $*$ and $**$ from these two definitions both impose the same $\text{Good}\{ \frac{1}{2} \sharp(\Phi) \}$ constraint on Υ and $\neg\Upsilon$.)

It is infeasible, however, for a sentence and its negation to both satisfy the same goodness constraint. This completes Theorem 5.9's proof-by-contradiction because the initial assumption that S was inconsistent has led to an infeasible conclusion. \square

Our next definition will help formalize a useful cousin of Theorem 5.9.

Definition 5.10 A Generic Configuration of ξ will be called **0-Stable** when every particular $\theta \in \text{RE-Class}(\xi)$ satisfies the invariant of $***$. (This invariant is strictly weaker than its counterparts $*$ and $**$ in Definitions 5.1 and 5.3.)

******* If Υ is a Δ_0^ξ theorem derived from the axiom system $\theta \cup B^\xi$ via a proof p whose length satisfies $\text{Log}(p) \leq \sharp(\theta) + 1$, then Υ is true under the Standard-M model.

Theorem 5.11 *If the configuration ξ is 0-stable then $B^\xi + \text{SelfCons}^0(B^\xi, d)$ is a self-justifying formalism.* (Appendix C shows this result applies also to E-stable and A-stable configurations.)

Theorem 5.11's proof is similar to Theorem 5.9's proof. The difference between these two propositions is that Theorem 5.11 has reduced SelfCons's superscript from 1 to 0, so that its hypothesis can encompass a theoretically broader set of applications. (The Appendix C summarizes how Theorem 5.9's proof can be easily modified to also prove Theorem 5.11.)

¹⁰ Let L denote the minimum value for $\text{Log}\{ \text{Max}[p, q, x, y] \}$ for a tuple (p, q, x, y) satisfying Equation (23). Then by definition, Φ satisfies $\text{Good}(L - 1)$ but not $\text{Good}(L)$. From Definition 4.4, this establishes the validity of Equation (24) (because the minimal $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ satisfying sentence (23) has $\text{Log}\{\text{Max}[\bar{p}, \bar{q}, \bar{x}, \bar{y}]\} = L$.)

Remark 5.12 Theorems 5.9 and 5.11 should clarify the nature of [59]–[68]’s formalisms. This is because proofs-by-contradictions are notorious in the mathematical literature for being confusing. They should be simplified whenever possible. This has been done mainly through Theorem 5.9’s short proof. (It applies to three of Appendix D’s four examples of generic configurations, and Theorem 5.11 applies to Appendix D’s fourth example.) Furthermore, Section 6 will show how more elaborate self-justification systems can verify all Peano Arithmetic’s Π_1^ξ theorems.

6 Four Further Meta-Theorems

We need one preliminary lemma before exploring how strong self-viewing logics may become before they cross the inevitable boundary between self-justification and inconsistency, implied by Gödel’s Theorem.

Lemma 6.1 *Let ξ denote a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, and θ^\bullet denote an r.e. set of Π_1^ξ sentences, each of which holds true in the Standard-M model. Let ξ^\bullet denote a 5-tuple that differs from ξ in that its base axiom system is $B^\xi \cup \theta^\bullet$ (rather than B^ξ). These conditions imply that ξ^\bullet is a generic configuration, and it will satisfy the following four invariants:*

- i *If ξ is 0-stable then ξ^\bullet will also be 0-stable .*
- ii *If ξ is A-stable then ξ^\bullet will also be A-stable .*
- iii *If ξ is E-stable then ξ^\bullet will also be E-stable .*
- iv *If ξ is EA-stable then ξ^\bullet will also be EA-stable .*

Lemma 6.1’s proof is fairly straightforward. It has been placed in the Appendix B. This section will use Lemma 6.1 to prove four meta-theorems that are consequences of its formalism.

Definition 6.2 Let ξ again denote a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, and θ denote some r.e. set of Π_1^ξ sentences (which are not required to be true under the Standard-M model). For the cases where k is either 0 or 1, the symbol $G_k^\xi(\theta)$ will denote the following axiom system:

$$G_k^\xi(\theta) = \theta \cup B^\xi \cup \text{SelfCons}^k\{ [\theta \cup B^\xi], d \} \quad (25)$$

Also when $k = 0$ or 1, the function G_k^ξ (which maps θ onto $G_k^\xi(\theta)$) is called **Consistency Preserving** iff $G_k^\xi(\theta)$ is assured to be consistent whenever all the sentences in θ are true under the Standard-M model.

We emphasize consistency preservation is unusual in logic. This is because $G_k^\xi(\theta)$ comes from adding a self-justifying axiom to an initially consistent formalism $B^\xi + \theta$, and

the Second Incompleteness Theorem demonstrates that sufficiently powerful formalisms are simply incompatible with such an axiom. However, there will be four specialized paradigms, defined in Appendix D, that are exceptions to this rule. They will be related to our next result:

Theorem 6.3 *The function G_1^ξ shall satisfy Definition 6.2's consistency preservation property when ξ is EA-stable. Likewise the function G_0^ξ will be consistency preserving when ξ is any one of A-stable, E-stable or 0-stable. (Thus in each case, $G_k^\xi(\theta)$ will be consistent when all the sentences in θ are true in the Standard-M model.)*

Proof. It will be convenient for our proof to use a dot-style notation, analogous to Lemma 6.1's terminology. Thus,

1. θ^\bullet denotes any r.e. set of Π_1^ξ sentences *that are each true* under the Standard-M model.
2. ξ^\bullet is the tuple $(L^\xi, \Delta_0^\xi, B^\xi \cup \theta^\bullet, d, G)$. (It differs from ξ by replacing ξ 's base axiom system of B^ξ with $B^\xi \cup \theta^\bullet$.)

Part-iv of Lemma 6.1 indicates the EA-stability of ξ implies the EA-stability of ξ^\bullet . Moreover Theorem 5.9 applies to all EA-stable configurations, including ξ^\bullet . Thus, $G_1^\xi(\theta^\bullet)$ is consistent because ξ is EA-stable and all of θ^\bullet 's sentences are true in the Standard-M model. This proves Theorem 6.3's first claim.

An almost identical proof, where Theorem 5.11 simply replaces Theorem 5.9 as the central self-justifying engine, will corroborate Theorem 6.3's second claim. Thus, G_0^ξ is consistency preserving when ξ is one of A-stable, E-stable or 0-stable. \square

Remark 6.4 Most generic configurations ξ will not satisfy Theorem 6.3's hypothesis because $G_k^\xi(\theta)$ will typically be inconsistent, irregardless¹¹ of whether or not all of θ 's axioms are valid under the Standard-M model. The significance of Theorem 6.3 is that it shows that some outlying exceptions to this general rule do prevail when ξ satisfies one of Theorem 6.3's four stability conditions. These exceptions are related to the 3-page abbreviated philosophical discussion that will appear later in Appendix F. They will assure that Theorem 6.3's formalisms of $G_1^\xi(\theta)$ and $G_0^\xi(\theta)$ are *always consistent* whenever θ 's axioms are valid in the Standard-M model.

Our next definition will enable self-justifying formalisms to prove the Π_1^ξ theorems of any consistent r.e. axiom system that uses L^ξ 's language.

Definition 6.5 Let ξ denote a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$. Let \mathcal{B} denote any recursive axiom system whose language is an extension of L^ξ . For an arbitrary

¹¹Conventional generic configurations ξ will satisfy the Hilbert-Bernays derivability conditions [16, 15]. Their $G_k^\xi(\theta)$ will thus be automatically inconsistent because of a Gödel-like diagonalization argument.

deduction method \mathcal{D} (which may be possibly different from ξ 's deduction method d), let $\text{Prf}_{\mathcal{B}}^{\mathcal{D}}(\lceil \Psi \rceil, q)$ denote a Δ_0^{ξ} formula indicating q is a proof of the theorem Ψ from axiom system \mathcal{B} , using deduction method \mathcal{D} . Then the **Group-2 Schema** for $(\mathcal{B}, \mathcal{D})$ is defined as an infinite set of axioms that includes one instance of (26)'s axiom for each Π_1^{ξ} sentence Ψ .

$$\forall q \{ \text{Prf}_{\mathcal{B}}^{\mathcal{D}}(\lceil \Psi \rceil, q) \longrightarrow \Psi \} \quad (26)$$

Comment about this Notation: The Definition 6.5 had called (26) a “Group-2 Schema” so as to keep our terminology consistent with [59, 61, 64, 66]'s notation.

Theorem 6.6 *Let ξ denote any arbitrary generic configuration, and $(\mathcal{B}, \mathcal{D})$ denote any pair consisting of an axiom system and a deduction method (which, once again, are allowed to be different from ξ 's deduction method and axiom system). Then if all of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} theorems are true in the Standard-M model, the following two invariants will hold:*

- i *If ξ is EA-stable then there will exist an r.e. self-justifying system that can prove all of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} theorems and recognize its own $\text{Level}(1^{\xi})$ consistency.*
- ii *Likewise, if ξ is one of A-stable, E-stable or 0-stable, there will exist an r.e. self-justifying system that can confirm all of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} theorems and which can recognize its own $\text{Level}(0^{\xi})$ consistency.*

Proof. Theorem 6.6 follows from Theorem 6.3. Thus let θ denote the set of all Π_1^{ξ} sentences that are members of Definition 6.5's Group-2 schema. Then every one of θ 's Group-2 axioms must be true under the Standard-M model (because the hypothesis of Theorem 6.6 indicated all of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} theorems are true in this model). Hence, Theorem 6.3 implies:

1. $G_1^{\xi}(\theta)$ is consistent when ξ is EA-stable.
2. $G_0^{\xi}(\theta)$ is consistent when ξ is one of A-stable, E-stable or 0-stable.

Since $G_0^{\xi}(\theta)$ and $G_1^{\xi}(\theta)$ are self-justifying systems that prove all of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} theorems, Items 1 and 2 will substantiate Theorem 6.6's two claims. \square

An awkward aspect of Definition 6.5's “Group-2” schema is that it employs an infinite number of instances of (26)'s Group-2-like axiom sentences. It turns out that this Group-2 scheme can be compressed into a *single axiom sentence*, if one is willing to settle for a slightly diluted variant of $(\mathcal{B}, \mathcal{D})$'s Π_1^{ξ} knowledge. To formalize this concept, the following notation shall be used:

1. $\text{Check}^\xi(t)$ will denote a Δ_0^ξ formula that checks to see whether t represents the Gödel number of a Π_1^ξ sentence.
2. $\text{Test}^\xi(t, x)$ will denote any Δ_0^ξ formula where (27)'s invariant is true under the Standard-M model for every Π_1^ξ sentence Ψ simultaneously. There are infinitely many different Δ_0^ξ formulae that can serve as $\text{Test}^\xi(t, x)$ predicates satisfying this condition. (Example 6.7 will illustrate one such encoding of a $\text{Test}^\xi(t, x)$ predicate.)

$$\Psi \longleftrightarrow \forall x \text{ Test}^\xi(\ulcorner \Psi \urcorner, x) \quad (27)$$

The expression (28) will be called a **Global Simulation Sentence** for representing $(\mathcal{B}, \mathcal{D})$ via ξ . Its $\text{Test}^\xi(t, x)$ clause essentially allows ξ to simulate the Π_1^ξ knowledge of $(\mathcal{B}, \mathcal{D})$'s set of theorems.

$$\forall t \forall q \forall x \{ \ulcorner \text{Prf}_\mathcal{B}^D(t, q) \wedge \text{Check}^\xi(t) \urcorner \longrightarrow \text{Test}^\xi(t, x) \} \quad (28)$$

Example 6.7 For any generic configuration $\xi = (L^\xi, \Delta_0^\xi, B^\xi, d, G)$, let $\text{NegPrf}^\xi(t, x)$ denote a Δ_0^ξ formula specifying that t is a Π_1^ξ sentence and that x is a proof under d 's deduction method of the Σ_1^ξ sentence that represents t 's negation. Also, let $\text{Test}_0^\xi(t, x)$ be defined as follow:

$$\text{Test}_0^\xi(t, x) =_{\text{def}} \neg \text{NegPrf}^\xi(t, x) \quad (29)$$

For each Π_1^ξ sentence Ψ , it is easy to verify ¹² that the statement (30) is true under the Standard-M model.

$$\Psi \longleftrightarrow \forall x \text{ Test}_0^\xi(\ulcorner \Psi \urcorner, x) \quad (30)$$

The latter implies that (31) is a global simulation sentence for $(\mathcal{B}, \mathcal{D})$.

$$\forall t \forall q \forall x \{ \ulcorner \text{Prf}_\mathcal{B}^D(t, q) \wedge \text{Check}^\xi(t) \urcorner \longrightarrow \text{Test}_0^\xi(t, x) \} \quad (31)$$

We emphasize that there are countably infinite different examples of $\text{Test}_i^\xi(t, x)$ predicates that generate global simulation sentences and that statement (31) illustrates only one such example.

Definition 6.8 Let $(\mathcal{B}, \mathcal{D})$ denote any ordered pair whose set of Π_1^ξ theorems are true under the Standard-M model. Let $\text{Test}_1^\xi, \text{Test}_2^\xi, \text{Test}_3^\xi, \dots$ denote the set of Δ_0^ξ formulae where statement (27) is true under the Standard-M model for every Π_1^ξ sentences Ψ . Then TestList^ξ will denote a list of all these Test_i^ξ predicates. Also for each Test_j^ξ formula in TestList^ξ , the symbol $\mathbf{GlobSim}_\mathcal{B}^D(\xi, j)$ will denote the special version of (28)'s global simulation formalism that employs Test_j^ξ 's machinery.

¹²Part-3 of Definition 3.4 indicated that ξ 's base axiom system is " Σ_1^ξ complete". (It is thus able to prove all Σ_1^ξ sentences that are true in the true in Standard-M model, and it will likewise refute all Π_1^ξ sentences that are false.) The statement (29) then immediately implies that (30) must be true under the Standard-M model for every Π_1^ξ sentence Ψ .

Remark 6.9 A comparison between Definition 6.5's Group-2 schema with 6.8's global simulation sentences will reveal neither is strictly better than the other. Both have their own separate advantages. Thus, the attractive aspect about Definition 6.8's $\text{GlobSim}_B^D(\xi, j)$ sentence is that it is a finite-sized object that can simulate the infinite set of axioms associated with Definition 6.5's Group-2 schema. The accompanying drawback¹³ of a global simulation sentence is that the union of it with the base-axiom system B^ξ will typically be inadequate to prove every Π_1^ξ sentence that is a theorem of $(\mathcal{B}, \mathcal{D})$. Instead, in a context where Ψ is a Π_1^ξ theorem of $(\mathcal{B}, \mathcal{D})$, the sentence $\text{GlobSim}_B^D(\xi, j)$ will usually provide only enough *fragmented information* to prove the statement (32) (which is equivalent to Ψ under the Standard-M model).

$$\forall x \text{ Test}_j^\xi(\lceil \Psi \rceil, x) \quad (32)$$

While (32) may be insufficient to prove Ψ from B^ξ , it still (according to the statement (27)) has the desired property of being equivalent to Ψ under the Standard-M model. (This means that the knowledge of (32)'s truth is helpful, *even if it is unknown from B^ξ 's perspective* to be equivalent to Ψ .)

Our prior articles [59, 61, 64, 66] did not use Definition 6.8's global simulation formalism. They employed, instead, Definition 6.5's Group-2 axiom schema. Theorem 6.10 will be the analog of Theorem 6.6 for global simulation. It will be useful when one desires to compress all the information held by a Group-2 schema into a single finite-sized object.

Theorem 6.10 *Let ξ denote the generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, \mathcal{B} denote a recursively enumerable axiom system and \mathcal{D} denote any deduction method (which can be different than ξ 's deduction method d). Suppose that all the Π_1^ξ theorems generated by $(\mathcal{B}, \mathcal{D})$ are true under the Standard-M model. Then the following invariants do hold:*

- i *If ξ is EA-stable then for each j there exists a **finitized extension** β_j of B^ξ that recognizes its $\text{Level}(1^\xi)$ self-consistency and which contains the sentence $\text{GlobSim}_B^D(\xi, j)$.*
- ii *Likewise for each j , if ξ is E-stable, A-stable or 0-stable then there exists a finitized extension β_j of B^ξ that recognizes its $\text{Level}(0^\xi)$ self-consistency and which contains the sentence $\text{GlobSim}_B^D(\xi, j)$.*

Proof: Let θ denote the 1-sentence R-View of “ $\text{GlobSim}_B^D(\xi, j)$ ” formalized by Definition 6.8, and let β_j^1 and β_j^0 denote $B^\xi \cup \theta + \text{SelfCons}^1(B^\xi \cup \theta)$ and

¹³The chief difficulty arises essentially because a Π_1^ξ theorem of $(\mathcal{B}, \mathcal{D})$ may contain an arbitrarily long combination of bounded universal and bounded existential quantifiers. Thus, some generic configurations will have base axiom systems B^ξ that are so weak that their combination with (28)'s Global Simulation Sentence is insufficient to prove the validity of (27)'s equivalence statement for all Π_1^ξ sentences Ψ simultaneously. In particular, such proofs will often be infeasible when Ψ 's sequence of bounded quantifiers has a length greatly exceeding the length of the Gödel encoding for (28)'s global simulation statement.

$B^\xi \cup \theta + \text{SelfCons}^0(B^\xi \cup \theta)$, respectively. These axiom systems correspond to the objects that Definition 6.2 had called $G_1^\xi(\theta)$ and $G_0^\xi(\theta)$.

Theorem 6.10's hypothesis indicates all the Π_1^ξ theorems of $(\mathcal{B}, \mathcal{D})$ are true under the Standard-M model. Thus, it follows that θ is also true in the Standard-M model. Hence Theorem 6.3 implies that $\beta_j^1 = G_1^\xi(\theta)$ is a consistent system satisfying Theorem 6.10's claim (i). Likewise, Theorem 6.3 implies $\beta_j^0 = G_0^\xi(\theta)$ satisfies Theorem 6.10's second claim. \square

Remark 6.11 Theorems 6.6 and 6.10 raise a fascinating question: *Is the trade-off between these formalisms needed?* That is, can self-justifying systems use only a finite number of added axioms beyond those lying in ξ 's base system of B^ξ and also duplicate all $(\mathcal{B}, \mathcal{D})$'s Π_1^ξ theorems in a pure sense (i.e. without simulation) ? We will return to this topic in Appendix G.

REFLECTION PARADIGMS : Our last goal is to show how self-justifying systems support *unusually* strong reflection principles. Let $\text{Reflect}_{\alpha,d}(\Psi)$ denote (33)'s statement when Ψ is a sentence with Gödel number $[\Psi]$, and (α, d) denotes an axiom system and deduction method.

$$\forall p \quad [\text{Prf}_{\alpha,d}([\Psi], p) \Rightarrow \Psi] \quad (33)$$

Löb's Theorem [15, 25, 42] implies that conventional systems (α, d) , possessing at least Peano Arithmetic's strength, are unable to prove $\text{Reflect}_{\alpha,d}(\Psi)$ except for in the degenerate cases where they can prove Ψ .

Moreover, it is easy to generalize Löb's Theorem (via say [61]'s Theorem 7.2) so that a wide class of formalisms α , weaker than Peano Arithmetic, are also unable to prove $\text{Reflect}_{\alpha,d}(\Psi)$ for all Π_1^ξ sentences Ψ simultaneously.

The intuition behind this generalization is quite simple. Let \mathcal{U} denote a Π_1^ξ encoding for the classic Gödel sentence declaring: “*There is no proof of me from the axiom system α using d 's deduction method*”. Then [61]'s Theorem 7.2 uses a very routine diagonalization argument to show most formalisms α will be inconsistent if they prove $\text{Reflect}_{\alpha,d}(\mathcal{U})$'s statement.

Our next theorem will show, surprisingly, that the preceding limitation is much less stringent than it may initially appear to be. This is because $\text{Level}(1^\xi)$ self-justifying axiom systems are capable of proving very close analogs to (33)'s impermissible

reflection principle, using a “translational” methodology.

Thus, let T denote an algorithm that maps a Π_1^ξ sentence Ψ onto a “translated” sentence Ψ^T that is equivalent to Ψ under the Standard-M model *and which is also written in a Π_1^ξ format*. (See footnote¹⁴ for why it is absolutely imperative that **both these** requirements be included in T ’s definition.) Also, let $\text{Reflect}_{\alpha,d}^T(\Psi)$ denote the translational modification of (33)’s reflection principle that replaces Ψ with Ψ^T .

$$\forall p \ [\text{Prf}_{\alpha,d}(\ulcorner \Psi \urcorner, p) \Rightarrow \Psi^T] \quad (34)$$

Theorem 6.12 *Let ξ denote the EA-stable configuration of $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, and let $\alpha = B^\xi + \text{SelfCons}^1(B^\xi)$ denote ξ ’s corresponding Level(1) self-justifying axiom system. Then there will exist a translation methodology T where α can prove the validity of (34) for all its Π_1^ξ sentences simultaneously.*

Proof: Let us use Example 6.7’s notation. It observed that Ψ ’s Π_1^ξ statement was equivalent under the Standard-M model to “ $\forall x \text{Test}_0^\xi(\ulcorner \Psi \urcorner, x)$ ”. Thus, let us view T as being a mapping of the first sentence onto the second.

Our proof of Theorem 6.12 will next use the following observations:

1. The non-existence of a proof of $\neg \Psi$ from $B^\xi + \text{SelfCons}^1(B^\xi)$ trivially implies the non-existence of a proof of the same theorem from B^ξ (because the latter axiom system is simply a subset of the former).
2. Moreover, Example 6.7’s notation treats “ $\forall x \text{Test}_0^\xi(\ulcorner \Psi \urcorner, x)$ ” as being equivalent to the declaration that no proof of $\neg \Psi$ from B^ξ exists.

Hence, α can prove (34)’s statement by noting that p ’s proof of a Π_1^ξ sentence Ψ implies (via α ’s SelfCons^1 axiom) the non-existence of a proof of $\neg \Psi$, which (via Items 1 and 2) implies “ $\forall x \text{Test}_0^\xi(\ulcorner \Psi \urcorner, x)$ ” \square .

Remark 6.13 : Theorem 6.12 and the statement (34)’s Translational Reflection Principle may possibly be useful devices in unraveling some of the mystery that has enshrouded Gödel’s Second Incompleteness Theorem, since its inception. This is partly because Gödel was explicitly uncertain about the generality of the Second Incompleteness Theorem in his initial 1931 seminal paper [14] about this subject. His centennial paper about Incompleteness thus included the following quite poignant caveat:

¹⁴ Part-3 of Definition 3.4 indicated that generic configurations are Σ_1^ξ complete. Our requirement that Ψ^T must have a Π_1^ξ format thus causes T to gain much added meaning. This is because the axiom system B^ξ will then automatically disprove Ψ^T whenever it is false under the Standard-M model. (Thus, T ’s mapping of Ψ onto Ψ^T gains much significance when Ψ and Ψ^T do rest on the same Π_1^ξ level of the arithmetic hierarchy.)

- : “It must be expressly noted that Theorem XI (i.e the Second Incompleteness Theorem) represents no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof by finite means, and *there might conceivably be finite proofs* which cannot be stated in ... ”

Some of the issues that troubled Gödel in the statement • can perhaps be partially resolved if one compares the reflection principles of sentences (33) and (34). This is because (33) is probably unnecessary to explain how thinking beings can appreciate their Π_1^ξ theorems when Theorem 6.12’s specialized logics can, instead, use the fact that its Π_1^ξ sentences satisfy at least (34)’s modified reflection principle. Thus, some of the mystery surrounding the Second Incompleteness Effect can be clarified when one notices that (34)’s translational reflection principle is a useful precept, that was shown by Theorem 6.12 to be technically unrelated to Gödel’s observation that no reasonable formalism can prove (33)’s *purist* principle for all Π_1^ξ sentences simultaneously.

Remark 6.14 Theorem 6.12 is also significant because it explains how its specialized logics can grapple with a Π_1^ξ encoded Gödel sentence \mathcal{U} which asserts “*There is no proof of me*”. This issue is challenging because routine constructions, such as [61]’s Theorem 7.2, demonstrate that no natural logic can verify statement (33)’s validity for all Π_1^ξ sentences Ψ (on account of the well-known syllogism posed by \mathcal{U} ’s Gödel sentence). Theorem 6.12 constructs, however, a reply to this challenge. This is because its self-justifying systems do surprisingly prove, without difficulty ¹⁵, the validity of (34)’s *translated modification* of (33)’s unobtainable Π_1^ξ styled variant of a reflection principle.

Remark 6.15 We encourage the reader to examine the work of Beklemishev, Kreisel-Takeuti and Verbrugge-Visser [5, 6, 7, 24, 52, 56] to see alternative reflection principles and their uses. (The constraint on proof-length, by Verbrugge-Visser, is certainly one alternative to Theorem 6.12’s machinery. Likewise, Kreisel-Takeuti’s Second Order Logic CFA reflection is another alternative, although it will not ¹⁶ generalize to first-order logics.) One complicating aspect of our Theorem 6.12’s reflection method is that Appendix E proves it becomes inoperable when an axiom system is sufficiently conventional to satisfy Gödel’s Second Incompleteness Theorem. In essence, Theorem 6.12’s translational

¹⁵Since Ψ and Ψ^T are equivalent under the Standard-M Model *but not also equivalent* from the perspective of the system $\alpha = B^\xi + \text{SelfCons}^1(B^\xi)$, the conventional contradictions, produced by (33)’s reflection principle, disappear when it is replaced by (34). (Thus, there is no danger that α could use (34)’s reflection principle to prove an analog of \mathcal{U} ’s forbidden Gödel sentence.)

¹⁶Kreisel-Takeuti indicate on page 25 of [24] that CFA’s reflection principle for a first order formula “A” infers the validity of *only the relativized formula* “ A^N ” from A’s proof. Also, their proof predicate is similarly relativized. Thus CFA’s second-order logic reflection principle, while fascinating, does not generalize to first-order logic environments.

reflection principle is a specialized methodology, intended for logics using Definition 5.7's Level(1) style of self-justification.

Remark 6.16 The preceding discussion clearly shows self-justifying logics are tempting. At the same time, it is necessary to be very cautious because there are also two fundamental barriers limiting such results:

- a The first is the Theorem 2.1 arising from the joint work of Pudlák, Solovay, Nelson and Wilkie-Paris [26, 33, 44, 58]. It showed no reasonable system recognizing successor as a total function can verify its own Hilbert consistency. Also, Willard [62, 67] established analogous results under semantic tableaux consistency for systems recognizing multiplication as a total function. Thus, each effort to evade the Second Incompleteness Theorem must encounter robust barriers.
- b A second issue is that Definition 5.7's "SelfCons" "*I am consistent*" axiom sentence is less than ideal because it causes axiom systems to produce essentially a 1-line proof of their own consistency. Such an excessively compressed proof corresponds more closely to an axiom system formulating an *instinctive faith* in its own consistency (rather than it supporting a full-length proof-justification of this fact).

Part of the reason self-justifying systems are of interest, despite these limitations, is that they illustrate how some formalisms *are compatible* with at least an *instinctive faith* in their own self-consistency. (This compatibility issue is non-trivial because Item (a) implies there are many circumstances where a generalization of the Second Incompleteness Theorem will make it infeasible for a formalism to satisfy *both* Parts (i) and (ii) of Section 1's definition of Self-Justification.) Moreover, three of Appendix D's four sample self-justifying configurations, called ξ^* , ξ^{**} and ξ^R , will be Type-A systems that recognize addition as a total function. These configurations will thus possess the following three significant finitized features:

1. They will be able to construct the entire infinite set of integers *by finite means* because they recognize addition as a total function.
2. For any r.e. logical configuration $(\mathcal{B}, \mathcal{D})$, it will be possible to develop a 1-sentence *finitized* extension for the base axiom systems of any of the configurations of ξ^* , ξ^{**} and ξ^R , which deploy (28)'s Global Simulation Sentence to simulate the Π_1^ξ knowledge of $(\mathcal{B}, \mathcal{D})$. This means that some fully finite-sized extensions of the base-formalisms of ξ^* , ξ^{**} and ξ^R will contain a non-trivial amount of Π_1^ξ styled knowledge, since $(\mathcal{B}, \mathcal{D})$ can correspond to, say, Peano Arithmetic.
3. The key point is that a 1-sentence extension of an axiom system containing features (1) and (2) can formalize how a logic can possess an instinctive faith in its own

consistency via Theorem 6.10's explicitly *finitized* structure. (Moreover, Theorem 6.12's Translational Reflection Principle is applicable to Appendix D's generic configurations of ξ^* and ξ^{**} . It will thus imply that their single *finitized* Level-1 self-justifying axioms enable them to prove an *infinite number* of incarnations of (34)'s translational reflection principle, where each Π_1^* sentence Ψ is mapped onto one such unique instance.)

The contrast between Items 1-3's positive remarks about "finitized" cogitation with Items (a) and (b)'s opposing comments is obviously formidable. It is clearly preferable to view these positive results cautiously and treat them as being no more than boundary-case exceptions to the Second Incompleteness Theorem. The essential reason why these exceptions are of interest is that Gödel's famous centennial paper has implicitly raised the following puzzling issue:

How is it that Human Beings manage to muster the physical drive to think (and prove theorems) when the many generalizations of Gödel's Second Incompleteness Theorem assert conventional logics lack knowledge of their own consistency?

There will, of course, never be any perfect answer to the puzzle posed by # because philosophical paradoxes and ironical dilemmas never yield perfect answers. However, part of an imperfect answer to # is that Items 1-3 reply to Challenges (a) and (b) by formalizing how a thinking being can muster an approximate partial *instinctive faith* in its own self-consistency. (Moreover, the tight contrast between various generalizations of the Second Incompleteness Theorem [1, 3, 8, 10, 15, 16, 21, 25, 33, 37, 44, 46, 58, 62, 63, 67] with the self-justifying systems appearing in Appendixes D and G suggests that these come close to being maximal forms of feasible results.)

Our remaining discussion will consist of four optional sections, called Appendixes D, E, F and G, which can be skimmed, omitted or examined in any order the reader prefers. A summary of their contents is given below:

- I** The Appendix D provides four examples of generic configurations that utilize Theorems 5.9, 5.11, 6.3, 6.6 and 6.10. Its most prominent examples involve Equation (2)'s Type-A axiom systems where the deduction method is either semantic tableaux or a modified version of tableaux that permits a modus ponens rule for Π_1^ξ and Σ_1^ξ sentences.
- II** The Appendix E introduces a generalization of the Second Incompleteness Theorem which shows that Theorem 6.12's Translational Reflection Principle applies *only* to self-justifying logics. (It is thus fully inoperative for conventional logics. This may explain why Theorem 6.12's self-justifying systems are an interesting topic.)

- III** The Appendix F differs from the rest of this paper by having a philosophical slant. It will offer a 3-page summary about why we suspect Theorem 6.3’s self-justification formalism and Remark 6.16’s notion of “instinctive faith” are useful.
- IV** The Appendix G introduces a “Braced $^\xi(\Phi, j)$ ” construct and two new theorems that hybridize the methodologies of Theorems 6.6 and 6.10. These results will improve upon Theorem 6.6 because their self-justifying systems contain only a finite number of axiom-sentences beyond those lying in ξ ’s base formalism of B^ξ . They will improve upon Theorem 6.10 because they can prove the important Braced $^\xi(\Phi, j)$ subset of $(\mathcal{B}, \mathcal{D})$ ’s Π_1^ξ theorems in a full sense (rather than in Remark 6.9’s weaker simulated respect). Appendix G’s results are useful because *for arbitrary k* and for any of Appendix D’s four sample configurations, every Π_1^ξ theorem of $(\mathcal{B}, \mathcal{D})$ containing k or fewer bounded and unbounded quantifiers will be proven by its Theorem G.3 to be self-justifying in an undiluted pure sense. (This is because each such Π_1^ξ sentence, with fewer than k quantifiers, will lie in some fixed Braced $^\xi(\Phi, j)$ set — where solely the value of k determines the values for Φ and j .)

It is probably desirable to concentrate primarily on Theorems 5.9, 5.11, 6.3, 6.6, 6.10 and 6.12 during one’s first reading of this paper. This is because Appendixes A-G are less central than these core theorems, although their material does add several useful further perspectives to this subject.

7. Concluding Remarks

The research in this article has been a continuation of our prior research [59]-[68] that simultaneously has simplified, unified and extended the prior results. It has explored self-justification with a 3-part approach where:

1. Sections 4 and 5 introduced three different stem components that can be used to generate self-justifying systems. (These are the relatively simple Lemma 4.6 and the mathematically more sophisticated Theorems 5.9 and 5.11.)
2. Section 6 and Appendix G then generalized our initial stem-like theorems in the six different directions formalized by Theorem 6.3, 6.6, 6.10, 6.12, G.2 and G.3
3. Appendix D subsequently provided four examples of generic configurations that are applications of Section 6’s results.

This 3-part approach is very different from the methods used in our prior articles [59, 64, 66, 68]. The latter examined particular isolated applications in thorough detail (rather

than compartmentalize and separate the analysis into three stages). The virtue of this 3-stage analysis is it leads to many new theorems, in addition to unifying our prior results.

It is desirable to categorize the maximal generality and strongest form of boundary-case exceptions for the Second Incompleteness Theorem that are feasible because Gödel's centennial discovery beckons the scholarly community to sharpen their understanding of his 1931 landmark discovery, that has fundamentally reshaped mathematics.

It should be emphasized that our over-all research in [59] – [68] has spent an approximately equal effort in exploring generalizations of the Second Incompleteness Theorem [62, 63, 66, 67] and in examining its viable boundary-case exceptions [59, 61, 64, 65, 66, 68] (although the current article focused on the latter topic). This is because the Second Incompleteness Theorem is a starkly robust result that imposes sharp limits on how strong self-justifying systems may become.

Finally, we encourage the reader to take another brief glance at Remarks 6.13 – 6.16. They offer a brief summary of both the strengths and limitations of our chief results. They also explain how Theorem 6.12's reflection principle for Π_1^ξ styled theorems is a very unexpected result.

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Appendix A: The Π_1^ξ encoding for $\text{SelfCons}^k(\beta, d)$

This appendix will summarize how to formalize a Π_1^ξ encoding for Definition 5.7's $\text{SelfCons}^k(\beta, d)$ predicate. It will use the following notation:

1. $\text{Neg}^k(x, y)$, will denote a Δ_0^ξ formula indicating that x is the Gödel number of a Π_k^ξ sentence and that y represents the Σ_k^ξ sentence which is its logical negation.
2. $\text{Prf}_\beta^d(t, p)$ will denote a formula designating that p is a proof of theorem t from the axiom system β using the deduction method d .
3. $\text{ExPrf}_\beta^d(h, t, p)$ will denote that p is a proof (using d 's deduction method) of a theorem t from the union of the axiom system β with the added sentence whose Gödel number equals h .
4. $\text{Subst}(g, h)$ will denote Gödel's substitution formula — which yields TRUE when g is an encoding of a formula and h encodes a sentence that replaces all occurrence of free variables in g with a term of \bar{g} (that specifies g 's Gödel number).
5. $\text{SubstPrf}_\beta^d(g, t, p)$ will denote the hybridization of Items 3 and 4 that yields a Boolean value of TRUE when there exists an integer h satisfying $\text{Subst}(g, h)$ and $\text{ExPrf}_\beta^d(h, t, p)$.

It is easy to apply [61]'s methodologies to confirm Items 1-5 can be encoded as Δ_0^ξ formulae. Thus, Appendixes C and D of [61] explained how the theory of LinH functions [15, 23, 69] implied there existed Δ_0 encodings for formulae 1-4, and these Δ_0 encodings can be easily rewritten¹⁷ as Δ_0^ξ expressions. Equation (35) uses this information to formulate a Δ_0^ξ encoding for $\text{SubstPrf}_\beta^d(g, t, p)$'s graph. It is equivalent to “ $\exists h \{ \text{Subst}(g, h) \wedge \text{ExPrf}_\beta^d(h, t, p) \}$ ”, but Equation (35) is written in a Δ_0^ξ format — *unlike* the quoted expression.

$$\text{Prf}_\beta^d(t, p) \quad \vee \quad \exists h \leq p \{ \text{Subst}(g, h) \wedge \text{ExPrf}_\beta^d(h, t, p) \} \quad (35)$$

Using (35)'s Δ_0^ξ encoding for $\text{SubstPrf}_\beta^d(g, t, p)$, it is easy to encode $\text{SelfCons}^k(\beta, d)$ as a Π_1^ξ axiom-sentence. Thus, let $\Gamma^k(g)$ denote (36)'s formula, and let \bar{n} denote $\Gamma(g)$'s Gödel number.

$$\forall x \forall y \forall p \forall q \quad \neg \{ \text{Neg}^k(x, y) \wedge \text{SubstPrf}_\beta^d(g, x, p) \wedge \text{SubstPrf}_\beta^d(g, y, q) \} \quad (36)$$

Then “ $\Gamma^k(\bar{n})$ ” is a Π_1^ξ encoding for $\text{SelfCons}^k(\beta, d)$'s formalization of the statement + from Definition 5.7. Thus, $\Gamma^k(\bar{n})$ is encoded is as follows:

$$\forall x \forall y \forall p \forall q \quad \neg \{ \text{Neg}^k(x, y) \wedge \text{SubstPrf}_\beta^d(\bar{n}, x, p) \wedge \text{SubstPrf}_\beta^d(\bar{n}, y, q) \} \quad (37)$$

¹⁷This rewriting of conventional Δ_0 formulae into a Δ_0^ξ format is possible because Part 2 of Definition 3.4 indicated that two 3-way predicates of $\text{Add}(x, y, z)$ and $\text{Mult}(x, y, z)$ do encode addition and multiplication in a Δ_0^ξ styled form.

Reminder about Equation (37) : This sentence's definition for $\text{SelfCons}^k(\beta, d)$ does not assure (37) is true under the Standard-M model. Indeed for nearly all (β, d) , it will be false when $k \geq 2$. This is the reason that the study of $\text{SelfCons}^k(\beta, d)$, under Theorems 5.9 and 5.11, has focused on the cases where k equals 0 or 1. Moreover, the preceding construction did assure that $\text{SelfCons}^k(\beta, d)$ had a Π_1^ξ encoding because such an “*I am consistent*” axiom carries more meaning than a Π_2^ξ encoded axiom.

Appendix B: The Proof of Lemma 6.1

Lemma 6.1 is a crucial interim step used to verify each of Theorems 6.3, 6.6 and 6.10. Its proof will employ the following three straightforward observations:

Fact B.1 Lemma 6.1's hypothesis implies that ξ^\bullet is a generic configuration. (This is because it specified that ξ was a generic configuration and that all the Π_1^ξ sentences of θ^\bullet were true in the Standard-M model. Thus, ξ^\bullet must also be a generic configuration.)

Fact B.2 The associative identity of $\theta \cup (\theta^\bullet \cup B^\xi) = (\theta \cup \theta^\bullet) \cup B^\xi$ obviously holds. It implies a sentence Υ is a theorem of $\theta \cup (\theta^\bullet \cup B^\xi)$ if and only if it is a theorem of $(\theta \cup \theta^\bullet) \cup B^\xi$.

Fact B.3 Lemma 6.1's hypothesis directly¹⁸ implies $\sharp(\theta) = \sharp(\theta^\bullet \cup \theta)$.

The justification of claims (i)-(iv) are consequences of Facts B.1 through B.3. We will provide a detailed proof of only Claim (i) here because all four claims have similar proofs.

Proof of Claim (i) : The hypothesis of Claim (i) indicated that ξ was 0-stable. Therefore, it satisfies Definition 5.10's invariant of $***$. In a context where ϕ is a variable designating a r.e. set of Π_1^ξ sentences and Υ is a variable corresponding to a Δ_0^ξ sentence, the invariant $***$ can be rewritten in a quasi-rigorous form as :

$$\forall \phi \ \forall \Upsilon \quad \text{the below statement, called } \Psi_1(\phi, \Upsilon), \text{ is true} \quad (38)$$

$\forall p$ If Υ is a Δ_0^ξ theorem derived from the axiom system $\phi \cup B^\xi$ via a proof p , whose length satisfies $\text{Log}(p) \leq \sharp(\phi) + 1$, then Υ is true under the Standard-M model.

Since (38)'s universally quantified variable ϕ can designate any r.e. set of Π_1^ξ sentences, it may designate the object “ $\theta \cup \theta^\bullet$ ”, where θ^\bullet is the r.e. set of Π_1^ξ sentences defined

¹⁸The identity of $\sharp(\theta^\bullet) = \infty$ must be true because the hypothesis of Lemma 6.1 indicated that all the Π_1^ξ sentences in θ^\bullet are true under the Standard-M model. By Definition 4.4, this implies $\sharp(\theta) = \sharp(\theta^\bullet \cup \theta)$.

by Lemma 6.1's hypothesis (and θ is any second r.e. set of sentences). Thus, (38) directly implies :

$$\forall \theta \ \forall \Upsilon \quad \text{the below statement, called } \Psi_2(\theta, \Upsilon), \text{ is true} \quad (39)$$

$\forall p$ If Υ is a Δ_0^ξ theorem derived from the axiom system $(\theta \cup \theta^\bullet) \cup B^\xi$ via a proof p , whose length satisfies $\text{Log}(p) \leq \sharp(\theta \cup \theta^\bullet) + 1$, then Υ is true under Standard-M.

Facts B.2 and B.3 enable one to simplify (39)'s terms of $(\theta \cup \theta^\bullet) \cup B^\xi$ and $\sharp(\theta \cup \theta^\bullet)$ and thus to derive (40) as a consequence.

$$\forall \theta \ \forall \Upsilon \quad \text{the below statement, called } \Psi_3(\theta, \Upsilon), \text{ is true} \quad (40)$$

$\forall p$ If Υ is a Δ_0^ξ theorem derived from axiom system $\theta \cup (\theta^\bullet \cup B^\xi)$ via a proof p , whose length satisfies $\text{Log}(p) \leq \sharp(\theta) + 1$, then Υ is true under the Standard-M model.

We will now use Fact B.1's observation that ξ^\bullet is a generic configuration. The sentence (40) indicates this configuration satisfies Definition 5.10's invariant of $***$. Hence, ξ^\bullet is 0-stable. \square

Brief Comments about the Justifications of Claims (ii)-(iv): This appendix has omitted proving (ii)-(iv) for the sake of brevity. Their proofs are similar to Claim (i)'s proof. For instance, Claim (ii)'s proof differs from Claim (i)'s proof by having the 0-stability invariant in $***$ replaced by the A-stability invariant of $*$. This will cause the analogs of (38) – (40) to undergo the following two simple changes under Claim (ii)'s proof :

1. Υ will represent a Π_1^ξ (rather than a Δ_0^ξ) theorem-statement under the revised versions of Ψ_1 , Ψ_2 and Ψ_3 used in Claim (ii)'s proof
2. The requirement (in sentences (38)-(40) of Claim i's proof) that Υ be true in the Standard model is changed to the stipulation that Υ satisfies the $\text{Good}\{\frac{1}{2} \sharp(\phi)\}$ and $\text{Good}\{\frac{1}{2} \sharp(\theta)\}$ conditions under the revised forms of Ψ_1 , Ψ_2 and Ψ_3 used to prove Claim (ii).

Other minor adjustments in Claim (i)'s proof shall verify Claims iii and iv.

Appendix C: The Proof of Theorem 5.11

Our proof of Theorem 5.11 is a straightforward modification of Theorem 5.9's proof. It will be divided into two lemmas.

Lemma C.1. *Every generic configuration that is either E-stable or A-stable will automatically satisfy Definition 5.10's 0-stability condition.*

Proof. Lemma C.1 is a consequence of the “Special Note” appearing at the end of Definition 4.2. For every $N \geq 0$, it indicated that if Υ is a Δ_0^ξ sentence then $\text{Scope}_E(\Upsilon, N)$ is equivalent to Υ . This implies (via Definition 4.3) that if Υ is a $\text{Good}(N) \Delta_0^\xi$ formula then $\text{Scope}_E(\Upsilon, N)$ is automatically true under the Standard-M model.

The latter observation makes it easy to confirm Lemma C.1. This is because every Δ_0^ξ sentence is a Π_1^ξ and Σ_1^ξ statement. Hence, the application of the invariants $*$ and $**$ from Definitions 5.1 and 5.3, in the degenerate case where Υ is a Δ_0^ξ theorem, corroborates Lemma C.1's claim (by showing that Definition 5.10's invariant of $***$ does hold). \square

The remainder of this appendix will focus on Definition 5.10's 0-stability condition. (This is sufficient to justify Theorem 5.11 because Lemma C.1 showed all E-stable and A-stable configurations are 0-stable.)

Lemma C.2. *Let ξ denote a generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$ that is 0-Stable. Then the axiom system of $B^\xi + \text{SelfCons}^0(B^\xi, d)$ will be consistent (and hence self-justifying).*

Proof: It will be awkward to write repeatedly the expression “ $B^\xi + \text{SelfCons}^0(B^\xi, d)$ ” during our proof. Therefore, H will be an abbreviated name for this system. Our justification of Lemma C.2, is similar to Theorem 5.9's proof, except it replaces a $\text{Level}(1^\xi)$ form of self-justification with a $\text{Level}(0^\xi)$. It will thus be abbreviated and use the following notation:

1. $\text{Prf}_H(t, p)$ is a Δ_0^ξ formula specifying p is a proof of the theorem t from the axiom system H (using ξ 's deduction method of d).
2. $\text{Neg}^0(x, y)$ is a Δ_0^ξ formula indicating x is the Gödel encoding of a Δ_0^ξ sentence and y is a Δ_0^ξ sentence representing x 's negation.

Expression (41) denotes H 's $\text{Level}(0^\xi)$ self-justification axiom. It is encoded using Appendix A's methodology, similar to its counterpart used in Theorem 5.9's proof (i.e. Equation (21)).

$$\forall x \forall y \forall p \forall q \neg \{ \text{Neg}^0(x, y) \wedge \text{Prf}_H(x, p) \wedge \text{Prf}_H(y, q) \} \quad (41)$$

Our proof of Lemma C.2 will be a proof by contradiction. It will thus begin with the contrary assumption that H is inconsistent and have Φ denote (41)'s sentence. The inconsistency of H implies that Φ is false under the Standard-M model. Hence via Definition 4.4, we get:

$$\sharp(\Phi) < \infty \quad (42)$$

Equation (42) implies there exists a tuple $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ satisfying (43). (This is because such a $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ corroborates (42)'s implication that a counter-example to (41)'s sentence does exist.)

$$\text{Neg}^0(\bar{x}, \bar{y}) \wedge \text{Prf}_H(\bar{x}, \bar{p}) \wedge \text{Prf}_H(\bar{y}, \bar{q}) \quad (43)$$

The (p, q, x, y) satisfying (43) with minimum value for $\text{Log}\{\text{Max}[p, q, x, y]\}$ will additionally satisfy (44). (This observation follows from the analog of the Footnote 10 appearing in Theorem 5.9's proof. Thus, Section 5's Equations of (23) and (24) are the analogs of the current Equations (43) and (44). The Footnote 10 showed the particular (p, q, x, y) satisfying (23) with minimum value for $\text{Log}\{\text{Max}[p, q, x, y]\}$ satisfied (24). By the same reasoning, the minimal $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ satisfying (43) will satisfy (44).)

$$\text{Log}\{\text{Max}[\bar{p}, \bar{q}, \bar{x}, \bar{y}]\} = \sharp(\Phi) + 1 \quad (44)$$

Equations (43) and (44) shall bring Lemma C.2's proof-by-contradiction to its sought-after end. Thus, let Υ denote the Δ_0^ξ sentence specified by \bar{x} . Then $\neg \Upsilon$ corresponds to the Δ_0^ξ sentence denoted by \bar{y} . Equation (44) indicates that both Υ and $\neg \Upsilon$ have proofs such that the logarithms of their Gödel numbers are bounded by $\sharp(\Phi) + 1$. Using Definition 5.10's invariant of $***$, these facts establish ¹⁹ that both Υ and $\neg \Upsilon$ are true under the Standard-M model.

But it is impossible for a sentence and its negation to be both true. This finishes Lemma C.2's proof because the temporary assumption that H was inconsistent has led to a contradiction. \square

Theorem 5.11 is a consequence of the Lemmas C.1 and C.2 because Lemma C.2's formalism generalizes to all E-stable and A-stable configurations via Lemma C.1's reduction methodology.

¹⁹To apply Definition 5.10's invariant $***$ in the present setting, one simply sets θ 's R-View equal to Φ 's 1-sentence statement. Then $***$ implies that both Υ and $\neg \Upsilon$ must be true under the Standard-M model because both their proofs had lengths $\leq \sharp(\Phi) + 1$.

Appendix D: Applications and Examples

This appendix will illustrate four examples of generic configurations that satisfy Theorems 5.9 and 5.11 (and which therefore are self-justifying). It will be divided into three parts. Section D-1 will define our first example of self-justifying configuration, called ξ^* . It will use semantic tableaux deduction. Section D-2 will prove that ξ^* is EA-stable. Section D-3 will briefly sketch three additional examples of stable generic configurations.

It is likely preferable to examine Sections 3 – 6 before this appendix. However, Section D-1’s short 2-page discussion can be read quite easily either before or after Section 6.

D-1. Definition of the EA-Stable Configuration ξ^*

Our first example of an EA-Stable configuration, called ξ^* , will be defined in this section. Its deduction method will be semantic tableaux. Its base axiom system B^* will be a Type-A formalism, which treats addition but not multiplication as a total function (i.e. see Equation (2)).

The closest analog of B^* and ξ^* in our prior work appeared in Section 5 of [64]. (It differed from ξ^* partly because it did not use Definition 3.4’s unifying notation.) In [64], a function F was called **Non-Growth** iff $F(a_1, a_2, \dots, a_j) \leq \text{Maximum}(a_1, a_2, \dots, a_j)$ for all a_1, a_2, \dots, a_j . Six examples of non-growth functions are:

1. *Integer Subtraction* where “ $x - y$ ” is defined to equal zero in *the special case* where $x \leq y$,
2. *Integer Division* where “ $x \div y$ ” equals x when $y = 0$, and it equals $\lfloor x/y \rfloor$ otherwise,
3. *Root*(x, y) which equals $\lceil x^{1/y} \rceil$ when $y \geq 1$, and it equals x when $y = 0$.
4. *Maximum*(x, y),
5. *Logarithm*(x) = $\lfloor \text{Log}_2(x) \rfloor$ when $x \geq 2$, and zero otherwise.
6. *Count*(x, j) = the number of “1” bits among x ’s rightmost j bits.

These operations were called **Grounding Functions** in [64]. The term **U-Grounding Function** referred to a set of functions that included the Grounding operations plus the further primitives of addition and $\text{Double}(x) = x + x$. (The Double operation is helpful because it significantly enhances [64]’s linguistic efficiency²⁰.)

The symbol Δ_0^* will be the analog of Definition 3.1’s Δ_0^ξ construct under the U-Grounding function language. (It will be defined to be any formula in a U-Grounding

²⁰The symbol $\text{Double}(x)$ was technically unnecessary in [64]’s formalism because $x + x$ can encode $\text{Double}(x)$. However, its notation adds expressive power to [64]’s language because, for example, $\text{Double}(\text{Double}(\text{Double}(\text{Double}(x))))$ requires less memory space to encode than x added to itself 16 times.

language where all its quantifiers are bounded.) It is easy in this context to encode a Δ_0^* formula $Mult(x, y, z)$ for representing multiplication's graph. For instance, Equation (45) is one such Δ_0^* formula (which actually does not employ any bounded quantifiers):

$$[(x = 0 \vee y = 0) \Rightarrow z = 0] \wedge [(x \neq 0 \wedge y \neq 0) \Rightarrow (\frac{z}{x} = y \wedge \frac{z-1}{x} < y)] \quad (45)$$

Expression (45) is significant because Part 2 of Definition 3.4 indicated every generic configuration must have available some method to represent the graphs of addition and multiplication in a Δ_0^ξ styled format, similar to (45)'s paradigm. (Addition can be treated trivially because the U-grounding language possesses an addition function symbol.) The footnote ²¹ serves as a reminder about why these Δ_0^* encodings are needed. Our next goal is to define the generic configuration ξ^* that Section D-2 will prove is EA-stable.

Definition D.1. The language L^* of the generic configuration ξ^* will be built in a natural manner out of the eight U-grounding function operations, the usual atomic predicate symbols of “ = ” and “ \leq ”, and the three constant symbols K_0 , K_1 and K_2 (that define the integers of 0, 1 and 2). The other components of ξ^* 's configuration are defined below:

- i As previously noted, Δ_0^* is defined to represent the set of all formulae in L^* 's language, whose quantifiers are bounded in an arbitrary manner by terms employing the U-Grounding function symbols. (It will thus generate via Definition 3.1 the Π_n^* and Σ_n^* sentences of L^* .)
- ii The base axiom system B^* for ξ^* will be allowed to be any consistent set of $\Pi_1^{\xi^*}$ sentences that is capable of proving every Δ_0^* sentence that is valid in Standard-M. It will also include sentence (46)'s *very precise* ²² Π_1^* styled declaration that addition is a total function.

$$\forall x \quad \forall y \quad \exists z \leq x + y : \quad \{ \quad z = x + y \quad \} \quad (46)$$

- iii ξ^* 's deduction method will be the semantic tableaux method.
- iv ξ^* 's Gödelized method g for encoding a semantic tableaux proof can be essentially any natural method that satisfies the minor stipulation that at least $5J$ bits are required to encode a semantic tableaux proof that has J function symbols. This

²¹Section 3 explained during its discussion of Lemma 3.6 that every generic configuration ξ must have a means to encode the graphs of addition and multiplication as Δ_0^ξ formulae (visavis Parts 1 and 2 of Definition 3.4). This enabled Lemma 3.6's procedure to translate all of conventional arithmetic's Σ_j and Π_j formulae into equivalent Σ_j^ξ and Π_j^ξ expressions.

²² The two appearances of the term “ $x + y$ ” in sentence (46) may at first appear to be redundant. (This statement is equivalent to sentence (4)'s declaration that addition is a total function, which had avoided such redundancy.) The virtue of (46)'s format is that it is a Π_1^* styled statement, unlike (4)'s Π_2^* styled format. This sharpened Π_1^* perspective will help simplify some of our proofs.

stipulation is called the “**Conventional Tableaux Encoding Requirement**”. It is trivial ²³ to corroborate that all the usual methods for encoding semantic tableaux proofs satisfy this criteria. (The Appendix A of [64] provides one example of a possible tableaux encoding method. Any other natural mechanism for encoding tableaux proofs is equally suitable.)

Section D-2 will, interestingly, prove that ξ^* is EA-stable. This will imply (via Theorem 5.9) that $B^* \cup \text{SelfCons}^1\{B^*, d\}$ is self-justifying. Theorems 6.6, 6.10, G.2 and G.3 will, formalize, in this context, four different methods in which B^* can be extended to construct self-justifying formalisms that are able to prove Peano Arithmetic’s Π_1^* theorems.

Thus while self-justifying axiom systems contain unavoidable weaknesses, they also possess the nice feature that they are able to prove many of the useful theorems of mathematics.

D-2. Proof of the EA-Stability of ξ^*

This section will prove ξ^* is EA-stable and thus satisfies the paradigms of Theorems 5.9, 6.3, 6.6, 6.10 and 6.12. Our proof will be based on modifying some of the methodologies from [64], so that they become applicable to ξ^* . Many readers may prefer to omit examining both this part of Appendix D and Section D-3 because they are unnecessary for understanding the material in Section 6 and Appendixes E and F. (Our recommendation is that the latter material be read first.)

Our notation for defining a semantic tableaux proof in the next paragraph will be similar to the conventional definitions appearing in Fitting’s and Smullyan’s textbooks [12, 40]. It will employ [64]’s notation so that we can employ two of its lemmas during our analysis of semantic tableaux proofs.

In our discussion, Φ will be called a **Prenex-Level(m^*)** sentence iff it is a Π_m^* or Σ_m^* expression that satisfies the usual prenex requirement (that all its unbounded quantifiers lie in its leftmost part). If Φ is Prenex-Level(m^*) then **Reverse**(Φ) shall denote a second Prenex-Level(m^*) sentence that is equivalent to $\neg \Phi$ rewritten ²⁴ in

²³The Conventional Tableaux Encoding Criteria requires that the Gödel number of a semantic tableaux proof, with J function symbols, must be least as large as 32^J . It is clear that all the usual methods for generating the Gödel codes satisfy this criteria. This is because any proof that has J function symbols will contain at least $2J$ logical symbols and thus employ at least $5J$ bits.

²⁴For example, if Φ denotes “ $\forall x \exists y \psi(x, y)$ ” then **Reverse**(Φ) would be written as “ $\exists x \forall y \neg \psi(x, y)$ ”.

a Prenex Level(m^*) form. For a fixed axiom system α , its **Φ -Based Candidate Tree** will be defined to be a tree structure whose root is the sentence $\text{Reverse}(\Phi)$ and whose all other nodes are either axioms of α or deductions from higher nodes of the tree, via the rules 1–8 given below. (The symbol “ $\mathbf{A} \implies \mathbf{B}$ ” in rules 1–8 will mean that \mathbf{B} is a valid deduction from its ancestor \mathbf{A} in the germane deduction tree.)

1. $\Upsilon \wedge \Gamma \implies \Upsilon$ and $\Upsilon \wedge \Gamma \implies \Gamma$.
2. $\neg\neg\Upsilon \implies \Upsilon$. Other rules for the “ \neg ” symbol are: $\neg(\Upsilon \vee \Gamma) \implies \neg\Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \rightarrow \Gamma) \implies \Upsilon \wedge \neg\Gamma$, $\neg(\Upsilon \wedge \Gamma) \implies \neg\Upsilon \vee \neg\Gamma$, $\neg\exists v \Upsilon(v) \implies \forall v \neg\Upsilon(v)$ and $\neg\forall v \Upsilon(v) \implies \exists v \neg\Upsilon(v)$.
3. A pair of sibling nodes Υ and Γ is allowed when their ancestor is $\Upsilon \vee \Gamma$.
4. A pair of sibling nodes $\neg\Upsilon$ and Γ is allowed when their ancestor is $\Upsilon \rightarrow \Gamma$.
5. $\exists v \Upsilon(v) \implies \Upsilon(u)$ where u is a newly introduced “Parameter Symbol”.
6. $\exists v \leq s \Upsilon(v) \implies u \leq s \wedge \Upsilon(u)$ is the variation of Rule 5 for bounded existential quantifiers of the form “ $\exists v \leq s$ ”.
7. $\forall v \Upsilon(v) \implies \Upsilon(t)$ where t denotes a U-Grounded term. These terms may be any one of a constant symbol, a parameter symbol (defined by a prior application of Rules 5 or 6 to some ancestor of the current node), or a U-Grounding function-symbol with recursively defined inputs.
8. $\forall v \leq s \Upsilon(v) \implies t \leq s \rightarrow \Upsilon(t)$ is the variation of Rule 7 for a bounded quantifier such as “ $\forall v \leq s$ ”.

Let us say a leaf-to-root branch (in a candidate tree) is **Closed** iff it contains both some sentence Υ and its negation “ $\neg\Upsilon$ ”. Then a **Semantic Tableaux Proof** of Φ , from the axiom system α , is defined to be a Φ -Based Candidate Tree whose every leaf-to-root branch is closed.

It is next helpful to define the notion of a **Z-Based Deduction Tree**, in a context where Z represents an axiom system, typically different from the prior paragraph’s α . This object will be defined to be identical to a semantic tableaux proof, except for the following changes:

- i *Every node* in a Z -Based deduction tree must be either an axiom of Z or a deduction from a higher node of the tree via the rules 1–8. (This applies also to the root of a Z -Based deduction tree. It will store an axiom of Z in its root, unlike a semantic tableaux proof which had stored $\text{Reverse}(\Phi)$ in its root.)

- ii There will be no requirement that each leaf-to-root branch be closed in a Z -Based deduction tree. (Indeed, some branch will automatically not be closed if Z is consistent.)

Items (i) and (ii) make it apparent that Z -Based deduction trees are different from semantic tableaux proofs. It will turn out, nevertheless, that the study of Z -Based deduction trees will clarify the nature of semantic tableaux proofs.

Definition D.2. Let a and b denote two integers that are powers of 2 satisfying $a > b \geq 2$. Then an axiom system Z (employing L^* 's language) will be called a **Normed(a,b)** formalism iff:

1. All Z 's axioms are either Π_1^* or Σ_1^* sentences.
2. Each Π_1^* axiom of Z will satisfy Definition 4.3's $\text{Good}(\log_2 a)$ criteria, and each Σ_1^* axiom of Z will likewise satisfy $\text{Good}(\log_2 b)$.

Clarification about Definition D.2 : The “Normed(a,b)” concept (above) is obviously equivalent to the same-named notion appearing in Definition 4 of [64]. It uses, however, a different notation to make it compatible with Section 4's formalism. Thus, Item 2's assertion that the Π_1^* axiom $\forall v_1 \forall v_2 \dots \forall v_k \phi(v_1, v_2, \dots, v_k)$ satisfies $\text{Good}(\log_2 a)$ is equivalent to (47)'s statement. The $\text{Good}(\log_2 b)$ property of $\exists v_1 \exists v_2 \dots \exists v_k \phi(v_1, v_2, \dots, v_k)$ is, likewise, equivalent to (48).

$$\forall v_1 < a \ \forall v_2 < a \ \dots \forall v_k < a \quad : \quad \phi(v_1, v_2, \dots, v_k) \ . \quad (47)$$

$$\exists v_1 < b \ \exists v_2 < b \ \dots \exists v_k < b \quad : \quad \phi(v_1, v_2, \dots, v_k) \ . \quad (48)$$

Our interests in this notation will center around Fact D.3's invariant:

Fact D.3 . Let ξ^* denote Definition D.1’s generic configuration, and Z be an extension of ξ^* ’s base axiom system B^* which satisfies Definition D.2’s Normed(a, b) constraint. Then any Z –Based deduction tree T that has a Gödel number smaller than $(a/b)^4$ must contain at least one root-to-leaf branch, called σ , that is not “closed”. (In other words, this path σ will be contradiction-free, insofar as it does not contain both some sentence Ψ and its formal negation).

Proof: The justification of Fact D.3 is a direct consequence of the Lemmas 1 and 2 appearing in article [64] (see footnote ²⁵ for more details). \square

²⁵ A proof of Fact D.3 from first principles would be quite complicated because there are eight elimination rules employed by semantic tableaux deduction, each of which needs to be examined by such a proof’s umbrella formalism. Fortunately, we do not need provide such a complicated analysis here because a 4-page proof of the Lemmas 1 and 2 in Section 5.2 of [64] had already visited these issues. Thus, Fact D.3 turns out to be an easy consequence of these two lemmas after the following two straightforward issues are addressed:

1. Section 5.2 of [64] had defined the “U-Height” of a deduction tree to be the largest number of U-Grounding function symbols that appear in any of its root-to-leaf branches. Its Lemma 1 proved that every deduction tree with a U-Height $\leq \text{Log}_2 a - \text{Log}_2 b$ will contain at least one branch satisfying a condition, which [64] called “Positive(a, b)”. The Lemma 2 in [64] then showed that this Positive(a, b) property implies that the germane deduction tree must contain some branch that is contradiction-free. The combination of these two lemmas thus amounts to the establishing of the following rephrased hybridized statement:

- If a Z -based deduction tree has a U-Height $\leq \text{Log}_2 a - \text{Log}_2 b$, then some branch of it is contradiction-free (i.e. this branch cannot contain both some sentence Ψ and its negation).

2. Fact D.3 ’s hypothesis indicated the Gödel number g for its deduction tree satisfied the following conditions:

- I. $g \leq (a/b)^4$
- II. The U-Height of g ’s deduction tree is less than $\frac{1}{5} \text{Log}_2 g$. (This is simply because Fact D.3 presumes that the “Conventional Tableaux Encoding” methodology from Part-iv of Definition D.1 was used to encode g ’s Gödel number.)

Items I and II imply g ’s tree has a U-Height $\leq \text{Log}_2 a - \text{Log}_2 b$. The invariant • then implies this deduction tree has at least one branch that is contradiction-free (as Fact D.3 claimed). \square

We emphasize that the above justification of Fact D.3 is *much simpler* than a proof from first principles. The latter would require examining eight different tableaux elimination rules, as the detailed proofs of [64]’s Lemmas 1 and 2 actually did do.

We will now apply Fact D.3 to prove Theorem D.4. Its invariant will, interestingly, collapse entirely ²⁶ if one were to merely add a multiplication function symbol to the U-Grounding language. This is why our boundary-case exceptions to the semantic tableaux version of the Second Incompleteness allow a Type-A axiom system to recognize addition as a total function (but suppress a similar treatment of multiplication).

Theorem D.4. *The generic configuration ξ^* is both A-stable and E-stable.* (This implies many different self-justifying formalisms exist via Theorems 5.9, 6.3, 6.6, 6.10, 6.12, G.2 and G.3.)

Our proof of Theorem D.4 will separately show ξ^* is A-stable and E-stable.

Proof of ξ^* 's A-stability : Suppose for the sake of establishing a proof by contradiction that ξ^* was not A-stable. Then the constraint $*$ of Definition 5.1 would be violated by at least some $\theta \in \text{RE-Class}(\xi)$. This violation will cause the statement $+$ to be true for such a θ :

- $+$ There exists a semantic tableaux proof p of a Π_1^* theorem, called say Υ , from the axiom system of $\theta \cup B^\xi$ such that $\text{Log}(p) \leq \#(\theta) + 1$ and where Υ also fails to satisfy $\text{Good}\{\frac{1}{2} \#(\theta)\}$.

Let us recall that if Υ is Π_1^* then $\text{Reverse}(\Upsilon)$ is a Σ_1^* sentence equivalent to $\neg\Upsilon$. Thus, $\text{Reverse}(\Upsilon)$ will satisfy $\text{Good}\{\frac{1}{2} \#(\theta)\}$ criteria (simply because it has the opposite goodness property as Υ). Also, if Z denotes the axiom system of $\theta \cup B^\xi + \text{Reverse}(\Upsilon)$, it is easy to verify ²⁷ that Z 's axioms will satisfy the $\text{Normed}\{2 \#(\theta), \sqrt{2 \#(\theta)}\}$ criteria.

It is next helpful to observe that what is a proof from one perspective corresponds to being a deduction tree from a different perspective. Thus, Item $+$'s proof p of the

²⁶ The difficulty posed by multiplication can be easily understood when one compares two integers sequences x_0, x_1, x_2, \dots and y_0, y_1, y_2, \dots , defined as follows:

$$x_i = x_{i-1} + x_{i-1} \quad \text{AND} \quad y_i = y_{i-1} * y_{i-1}$$

It turns out that the faster growth rate of multiplication under the series y_0, y_1, y_2, \dots enables one to construct tiny Z -Based deduction trees T that violate the analog of Fact D.3's paradigm. (This is because such trees can have Gödel numbers smaller than $(a/b)^4$, while all their root-to-leaf branches can be simultaneously "closed" via contradictions.) This property of multiplication is analogous to Example 3.7's observations about how the differing growth rates of x_0, x_1, x_2, \dots and y_0, y_1, y_2, \dots are related to the threshold where the semantic tableaux version of Second Incompleteness Theorem can be evaded.

²⁷ The axiom system Z must satisfy $\text{Normed}\{2 \#(\theta), \sqrt{2 \#(\theta)}\}$ because:

1. The quantity $2 \#(\theta)$ is a valid first component for Z 's norming constraint because all the axioms of B^ξ are true in the Standard-M model and because Definition 4.4 implies all of θ 's axioms satisfy $\text{Good}\{\frac{1}{2} \#(\theta)\}$.
2. The quantity $\sqrt{2 \#(\theta)}$ is a valid second component for Z 's norming constraint because $\text{Reverse}(\Upsilon)$ is the only Σ_1^* sentence belonging to Z , and because $\text{Reverse}(\Upsilon)$ satisfies $\text{Good}\{\frac{1}{2} \#(\theta)\}$.

theorem Υ from the axiom system of $\theta \cup B^\xi$ corresponds to being a Z-based deduction tree, with Z representing the axiom system of $\theta \cup B^\xi + \text{Reverse}(\Upsilon)$. In this context, Item ++'s inequality of $\text{Log}(p) \leq \sharp(\theta) + 1$ implies²⁸ that p , viewed as a deduction tree for Z , satisfies the hypothesis of Fact D.3. Hence, Fact D.3 establishes that p must contain at least one contradiction-free root-to-leaf branch.

This last observation is all that is needed to confirm ξ^* 's A-stability, via a proof-by-contradiction. This is because the definition of a semantic tableaux proof implies every one of its root-to-leaf branches must end with a pair of contradicting nodes. However, the last paragraph showed p will not satisfy this required property, if ξ^* is not A-stable. Hence our construction has proven the A-stability of ξ^* by showing that otherwise an infeasible circumstance will arise. \square

Proof of ξ^* 's E-stability : A proof-by-contradiction will verify ξ^* is E-stable, analogous to the proof of its A-stability. Thus if ξ^* was not E-stable, then statement ++ would be true for some θ . (This is because at least one $\theta \in \text{RE-Class}(\xi)$ would then violate Definition 5.3's requirement of $**$.)

++ There exists a semantic tableaux proof p of a Σ_1^ξ theorem Υ from the axiom system $\theta \cup B^\xi$ such that $\text{Log}(p) \leq \sharp(\theta) + 1$ and Υ also fails to satisfy $\text{Good}\{\frac{1}{2} \sharp(\theta)\}$.

Item ++ implies $\text{Reverse}(\Upsilon)$ satisfies $\text{Good}\{\frac{1}{2} \sharp(\theta)\}$ (because $\text{Reverse}(\Upsilon)$ again has the opposite goodness property as Υ). Let Z now denote the formal axiom system of $\theta \cup B^\xi + \text{Reverse}(\Upsilon)$. The footnote²⁹ then uses reasoning similar to footnote 27 to show Z satisfies $\text{Normed}\{\sqrt{2 \sharp(\theta)}, 2\}$

As before via a simple change in notation, p 's semantic tableaux proof of Υ can be viewed as a deduction tree using Z 's axioms. Also as before, we may use the combination of the facts that Z is a $\text{Normed}\{\sqrt{2 \sharp(\theta)}, 2\}$ system and that Item ++ indicated $\text{Log}(p) \leq \sharp(\theta) + 1$ to deduce³⁰ that p is small enough to satisfy Fact D.3's hypothesis.

²⁸ Without loss of generality, we may assume that every non-trivial proof p satisfies $\text{Log}(p) \geq 64$ (since a string with fewer than 64 bits is too short to be a proof). Then the footnoted paragraph's $\text{Log}(p) \leq \sharp(\theta) + 1$ inequality trivially implies $p < 3^{\sharp(\theta)}$. In a context where Z is a $\text{Normed}\{2^{\sharp(\theta)}, \sqrt{2 \sharp(\theta)}\}$ axiom system, the latter inequality certainly implies p , viewed as a deduction tree for Z , has a small enough Gödel number to satisfy the hypothesis for Fact D.3. (This is because if one sets $a = 2^{\sharp(\theta)}$ and $b = \sqrt{2 \sharp(\theta)}$ then obviously $p < 3^{\sharp(\theta)} < 4^{\sharp(\theta)} = (a/b)^4$).

²⁹ The axiom system Z must satisfy $\text{Normed}\{\sqrt{2 \sharp(\theta)}, 2\}$ because:

1. The first component of its norming constraint can be set equal to $\sqrt{2 \sharp(\theta)}$ because $\text{Reverse}(\Upsilon)$ is a $\text{Good}\{\frac{1}{2} \sharp(\theta)\} \Pi_1^*$ sentence, and all Z 's other Π_1^* sentences satisfy more relaxed constraints.
2. The second component of Z 's norming constraint is satisfied by the constant of 2 because Definition D.2 implies this quantity is always permissible when Z contains no Σ_1^* axiom sentences.

³⁰ The proof that p is small enough to satisfy Fact D.3's hypothesis in the current E-stable case is

Hence once again, Fact D.3 implies that Z must contain at least one contradiction-free root-to-leaf branch. As before, the existence of this contradiction-free path violates the definition of a semantic tableaux proof and enables our proof-by-contradiction to reach its desired end. \square

Remark D.5 (*about Theorem D.4's significance*) : Part-ii of Definition D.1 indicated ξ^* 's base axiom of B^* was a Type-A formalism that recognized addition as a total function. This is significant because [60, 62, 67, 68] showed nearly all Type-M formalisms, including all the common axiomatizations for IS_0 , are unable to recognize their semantic tableaux consistency. Thus, the declaration that multiplication is a total function is *the trigger-point* causing³¹ the semantic tableaux version of the Second Incompleteness Theorem to become active. This threshold effect is significant because Theorem D.4, combined with Theorems 6.6, 6.10, G.2 and G.3, formalize *four different respects* in which Type-A self-justifying formalisms can prove all Peano Arithmetic's Π_1^* theorems (*after* multiplication's totality axiom is suppressed).

D-3. Three Further Examples of Stable Generic Configurations

Our second example of an EA-stable configuration is called ξ^{**} . It will be identical to ξ^* except that it will replace semantic tableaux with a stronger deduction method, which [64] called $\text{Tab}-U_1^*$. The latter is a revised version of semantic tableaux that permits a modus ponens rule to perform deductive cut operations on Π_1^* and Σ_1^* sentences. (The formal definition of $\text{Tab}-U_1^*$ deduction had appeared in [64]. It will be unnecessary to repeat here.)

The Section 5.3 of [64] noted $\text{Tab}-U_1^*$ has similar self-justification properties as conventional semantic tableaux. All the results that Section D-2 proved about ξ^* apply also to ξ^{**} , via their natural generalization under [64]'s $\text{Tab}-U_1^*$ deduction method. Thus, ξ^{**} is also EA-stable.

A key point is that there is a non-trivial distinction between ξ^* and ξ^{**} , despite the fact that they have similar technical qualities. This is because ξ^{**} contains a Level-1 modus ponens rule (unlike ξ^*). If it were infeasible to expand ξ^* into a broader ξ^{**} , then both formalisms could, perhaps, be easily dismissed as having negligible pragmatic

almost identical to Footnote 28's analysis of the A-stable case. Thus as in the earlier case, Item ++'s inequality of $\text{Log}(p) \leq \#(\theta) + 1$ trivially implies $p < 3^{\#(\theta)}$. Also, we may again assume that $\text{Log}(p) \geq 64$ (since a sequence with fewer than 64 bits cannot amount to a proof of any interesting fact under all normal coding conventions). An analog of Footnote 28's chain of inequalities will then allow us to conclude that p is small enough proof from a Normed $\{\sqrt{2^{\#(\theta)}}, 2\}$ system to satisfy the hypothesis for Fact D.3.

³¹We formally proved in [60, 62, 67, 68] that multiplication's totality property causes the semantic tableaux version of the Second Incompleteness Theorem to become active. The Example 3.7 summarizes the main intuition behind these results.

significance (since modus ponens is central to cogitation). However in a context where ξ^{**} does permit a Level-1 modus ponens rule, it is a tempting formalism (despite its limited modus ponens rule).

Unlike ξ^* and ξ^{**} , our third example of an EA-stable configuration, called ξ^- , will support an unlimited modus ponens rule. This will be possible because ξ^- 's language of L^- will be weaker than the languages of ξ^* and ξ^{**} . Thus L^- will include the six Grounding functions, but not the Growth functions of addition and doubling. It will thus treat addition and multiplication as 3-way atomic predicates, $\text{Add}(x, y, z)$ and $\text{Mult}(x, y, z)$, rather than as formal functions.

This perspective enabled ξ^- to support an evasion of the Second Incompleteness Theorem with an unlimited modus ponens rule present, in a context where the other four parts of its generic configuration are defined below:

1. The Δ_0^- class for ξ^- will be built in an essentially natural manner from the Grounding function set. It will thus include all formulae in L^- 's language, whose quantifiers are bounded in any arbitrary manner using the Grounding function primitives.
2. The base axiom system B^- of ξ^- will employ an infinite number of constant symbols, denoted as K_1, K_2, K_3, \dots where $K_1 = 1$ and where K_{i+1} is a power of 2 defined by the axiom of:

$$\text{Add}(K_i, K_i, K_{i+1}) \quad (49)$$

Thus, the combination of K_1, K_2, K_3, \dots with the Grounding function of subtraction allows the language L^- to encode the value of any arbitrary natural number (as Part 1 of Definition 3.4 had required). Essentially, ξ^- 's base axiom system of B^- can be any consistent r.e. set of Π_1^- sentences that includes (49)'s axiom schema and is able to prove every Δ_0^- sentence which is valid in the Standard-M model.

3. ξ^- 's deduction method can be any version of a classic Hilbert-style proof methodology. (Thus, it will include a modus ponens rule with no restrictions.)
4. ξ^- 's Gödelization method can be essentially any natural technique.

An interesting aspect of ξ^- is it can be proven to be EA-stable via an analog of Section D-2's treatment of ξ^* . Thus, Theorem 6.6 implies every axiom system α , whose Π_1^- theorems hold true in the Standard-M model, can be mapped onto an extension of ξ^- 's base axiom system that can recognize its own Hilbert consistency and prove α 's Π_1^- theorems. Except for minor changes in notation, this result represents a new way of proving [66]'s Theorem 3.

The self-justifying features of ξ^* , ξ^{**} and ξ^- are of interest primarily because the Second Incompleteness Theorem implies that they cannot be improved much further. This tight fit is summarized by Items 1-4.

1. The Theorem 2.1 (due to the combined work of Nelson, Pudlák, Solovay and Wilkie-Paris [26, 33, 44, 58]) implies no natural axiom system can prove Successor is a total function and recognize its own Hilbert consistency. This theorem thus explains why the presence of growth functions must be omitted from ξ^- 's base axiom system of B^- .
2. Moreover, [66] proved ξ^- 's method for evading the Second Incompleteness Theorem will collapse if one replaces Equation (49)'s "addition-based named sequence" of constant symbols K_1, K_2, K_3, \dots with a faster growing "multiplicative convention", where the constant symbols C_1, C_2, C_3, \dots are formally defined via (50)'s schema.

$$\text{Mult}(C_i , C_i , C_{i+1}) \tag{50}$$

Thus, [66] showed that there exists a Π_1^- sentence W (provable from Peano Arithmetic) such that no consistent system can simultaneously prove W , contain (50)'s axiom schema and prove the non-existence of proof of $0 = 1$ from itself. There is no space to prove it here, but a generalization of the Second Incompleteness Theorem implies the modification of ξ^- that replaces (49)'s axiom schema with (50)'s schema *is not even 0-stable*.

3. Similarly, [62, 67] proved that if ξ^* 's and ξ^{**} 's base axiom system of B^* was strengthened to include the assumption that multiplication was a total function then [64]'s two semantic tableaux evasions of the Second Incompleteness Theorem would both collapse.
4. Also, [63] proved that an analog of ξ^{**} 's evasion of the Second Incompleteness Theorem will collapse if its modus ponens rule was expanded to apply to either Π_2^* or Σ_2^* sentences.

The Item 3 is especially interesting because [65] proved [64]'s evasion of the Second Incompleteness Theorem was compatible with its formalism recognizing an infinitized generalization of a computer's floating point multiplication as a total function. Thus while the semantic tableaux formalisms of ξ^* or ξ^{**} are provably unable [62, 67] to recognize integer multiplication as a total function, their relationship to floating point multiplication is more subtle.

Our fourth example of an application of Section 6's theorems was stimulated by some insightful email we received from L. A. Kołodziejczyk [20] in 2005. It noted there existed a potential exponential gap between the lengths of semantic tableaux and Herbrand-style

proofs under some circumstances. Our earlier research [62] addressed a 1981 Paris-Wilkie open question [30] by generalizing some Adamowicz-Zbierski techniques [1, 3] to show a natural axiomatization of $I\Sigma_0$ satisfied the semantic tableaux version of the Second Incompleteness Theorem. In this context, Kołodziejczyk asked whether this would apply to all plausible axiomatizations for $I\Sigma_0$?

We replied in [68] to Kołodziejczyk’s stimulating question by distinguishing between Example 3.2’s Δ_0^A and Δ_0^R formulae and by using the Paris-Dimitracopoulos [28] translation algorithm for Δ_0 formulae. (The latter procedure was summarized earlier by Lemma 3.6. It demonstrated how to map classic arithmetic’s Δ_0^A formulae onto equivalent Δ_0^R formulae in the Standard-M model.) Our reply to Kołodziejczyk’s question, thus, employed this translation methodology to show that there existed an axiom system, called Ax-3, which proved the identical set of theorems as the more common Ax-1 and Ax-2 encodings of $I\Sigma_0$ and which possessed the following pair of quite fascinating contrasting properties:

- A** No consistent superset β of Ax-3’s set of axioms is capable of proving its own semantic tableaux consistency [68].
- B** In contrast, if “Herb” denotes the next paragraph’s Herbrand-styled deduction and if “SelfRef” denotes the sentence \bullet from Section 1, then $\text{Ax3} + \text{SelfRef}(\text{Ax-3}, \text{Herb})$ will be a self-justifying axiom system.

The intuition behind [68]’s proof of Items A and B can be easily summarized if we define a “Herbrandized-style” proof of a theorem Φ from an axiom system α as being an essentially 2-part structure where:

1. Each of α ’s axioms and also the sentence $\neg\Phi$ are first written as Skolemized expressions.
2. A propositional calculus proof is then used to show that some formal conjunction of instances of Item 1’s Skolemization schema has no satisfying truth assignment.

Such a formalism is different from the definition of a semantic tableaux proof (appearing in for example Fitting’s textbook [12]). This is because the latter replaces the use of Skolemization in Items 1 and 2 with an existential quantifier elimination rule. It turns out that this distinction enables some semantic tableaux proofs to be exponentially more compressed than their Herbrandized counterparts, as Kołodziejczyk observed [20, 21]. This fact enabled [68] to prove that Herbrandized and semantic tableaux proofs have the divergent properties summarized by Items A and B.

One reason Ax-3’s evasion of the Second Incompleteness Theorem is of interest is that $I\Sigma_0$ supports many more generalizations of the Second Incompleteness Theorem than evasions of it. Thus, Willard [62, 67, 68] proved that the semantic tableaux version

of the Second Incompleteness Theorem was valid for three different encodings of $\text{I}\Sigma_0$, and Adamowicz, Salehi and Zbierski have discussed in great detail [1, 3, 37] various Herbrandized generalizations of the Second Incompleteness Theorem for particular encodings of $\text{I}\Sigma_0$ and $\text{I}\Sigma_0 + \Omega_i$. Moreover, an added facet of [68]’s Ax-3 encoding for $\text{I}\Sigma_0$ is that most automated theorem provers use a particular variant of the Resolution method that causes [68]’s unusual methodology to apply also to them ³².

The reason for our interest in [68]’s results is that it represents a fourth example where the meta-theorems from Sections 5 and 6 can be useful. Thus, the footnote ³³ summarizes how a fourth type of generic configuration, called ξ^R , can be defined that both duplicates [68]’s main self-justification results under the above definition of Herb-deduction, as well as strengthens them. (In particular, ξ^R meets Theorem 5.11’s requirements, and self-justifying extensions of its Ax-3 system thus recognize their $\text{Level}(0^R)$ consistency.)

The properties of our four generic configurations of ξ^R , ξ^* , ξ^{**} and ξ^- are summarized by Table I. These configurations are listed in ascending order according to the strength of their deduction methods d . As their deduction methods increase in strength, these configurations have their ability reduced to recognize the totality of the addition and multiplication operations.

ξ^R is thus a Type Almost-M system that can prove multiplication is a total function (but which does not contain Equation (5)’s totality statement *as an axiom*). On the other hand, ξ^- uses a stronger Hilbert-styled deduction methodology, which is incompatible with treating the totality of addition or multiplication as either axioms *or as derived theorems*.

³²The main theorems in [68] generalize for resolution because Resolution-based theorem provers employ skolemization analogously to Herbrand deduction.

³³The discussion in [68] did not technically use Definition 5.3’s machinery to establish there existed an extension of its “Ax-3” encoding for $\text{I}\Sigma_0$ that could recognize its own Herbrand consistency. Its formalism, however, could be easily couched in terms of Definition 5.3’s machinery, if one uses a generic configuration ξ^R where

1. ξ^R ’s base language is the same as the usual language of arithmetic,
2. ξ^R ’s Δ_0^R sub-class is defined by Item (b) in Example 3.2,
3. ξ^R ’s base axiom system is [68]’s “Ax-3” system,
4. ξ^R ’s deduction method is either a Herbrandized styled-method or a Resolution system that relies upon Skolemization in a similar manner.
5. ξ^R ’s Gödel encoding scheme may be any such natural method.

This approach supports a stronger form of self-justification result than had appeared in [68]. This is because ξ^R can be proven to be E-stable (by a generalization of [68]’s analysis techniques). Thus, Theorem 5.11 implies that Ax-3 has a well-defined self-justifying extension that can recognize its own formalized $\text{Level}(0^R)$ consistency. (This self-justification result is stronger than [68]’s main theorem. The latter merely established that some extension of Ax-3 recognized the non-existence of a Herbrandized deduction of $0 = 1$ from itself.)

Each of Table I's rows ξ^R , ξ^{**} and ξ^- are maximal (in that an alternate row improves upon one column's measurement only when it is weaker from the perspective of another column). Only ξ^* is an exception to this rule: It is strictly weaker ³⁴ than ξ^{**} . This appendix has discussed ξ^* because it makes Theorem D.4's proof simpler (and also because semantic tableaux is a frequent topic in the logic literature).

Table I

| Name | Deduction Method | Type A | Almost M | Type M | Axiom Format | Self-Just Level |
|------------|---|-------------------|--------------------|------------------|------------------------|----------------------|
| ξ^R | Resolution and/or Herbrandized analogs | Yes ³⁵ | Yes | No | E-stable | Level (0^R) |
| ξ^* | Semantic Tableaux | Yes | No | No | EA-stable | Level (1^*) |
| ξ^{**} | Tab- U_1^* Deduction ³⁴ | Yes | No | No | EA-stable | Level (1^*) |
| ξ^- | Hilbert Deduction | No | No | No | EA-stable | Level (∞^-) |

The footnote ³⁵, attached to Table I's first row, explains why a caveat is attached to its first "Yes" entry. The theme of Table I is that self-justifying axiom systems have some nice redeeming features, although the Second Incompleteness Theorem clearly also imposes severe limits on their abilities. This point will be reinforced when Appendix E introduces a generalization of the Second Incompleteness Theorem, that shows Theorem 6.12's translational reflection principle is close to being a maximal feasible result, and when Appendix F discusses the epistemological significance of self justification.

³⁴ ξ^{**} employs a stronger deduction method than ξ^* because it allows a modus ponens rule for Π_1^* and Σ_1^* sentences to be added to semantic tableaux deduction (see [64] for the precise definition of this "Tab- U_1^* " modification of the semantic tableaux deductive method).

³⁵For the sake of simplicity, the Ax-3 system of [68] did not use either Equations (2) or (4)'s as axiom statements (since they were provable as theorems). All [68]'s results do, however, generalize when (2)'s statement about addition's totality is included as an axiom. Thus, it is appropriate to attach the designation of "Yes" with a caveat to the "Type-A" entry in Table I's first row. (This row is called "Resolution and/or Herbrandized analogs" because it applies to essentially any deduction scheme that relies upon Skolemization as an alternative to [12]'s semantic tableaux existential quantifier elimination rule.)

Appendix E: A Clarification of Theorem 6.12's Significance

It has been known since the time of Gödel that most conventional arithmetic axiom systems will satisfy the following two invariants:

1. They are *physically unable* to prove their own consistency
2. They are Σ_1 complete. This means they can formally prove any Σ_1 arithmetic sentence that holds true in the Standard-M model, and they can likewise refute any Π_1^ξ sentence that is false.

Let ξ denote any generic configuration of the form $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$. This appendix will use the term ξ –**Conventional** to describe any axiom system that satisfies analogs of the preceding conditions for generic configurations. Thus α is ξ –Conventional iff it satisfies the following two criteria:

- a. The axiom system α will be *unable* to verify its own consistency under ξ 's deduction method of d .
- b. The axiom system α will be an extension of ξ 's base axiom of B^ξ . Part-3 of Definition 3.4 will thus imply it is Σ_1^ξ complete. (Hence, α can formally prove any Σ_1^ξ sentence that holds true in the Standard-M model, and it can likewise refute any Π_1^ξ sentence that is false.)

This section will prove no analog of Equation (34)'s translational reflection principle is feasible for ξ –Conventional axiom systems. Thus, Theorem 6.12 must be close to being a maximal result, since it cannot plausibly be further extended to hold under conventional axiom systems.

Theorem E.1 (A New Type of Version of the Second Incompleteness Theorem): *There exists no ξ –Conventional axiom system α that can prove the validity of (51)'s Translational Reflection Principle for any translation-mapping T . (In other words, there exists no algorithm T that maps Π_1^ξ sentences Ψ onto alternate Π_1^ξ sentences Ψ^T , which are equivalent to Ψ in the Standard-M model and where α can verify (51)'s reflection principle for every Π_1^ξ sentence Ψ .)*

$$\forall p \quad [\text{Prf}_{\alpha,d} (\lceil \Psi \rceil , p) \Rightarrow \Psi^T] \quad (51)$$

Proof: It is easy to prove Theorem E.1 via a proof-by-contradiction. Thus consider the possibility that Theorem E.1's translational mapping T did exist. One can then easily select a Π_1^ξ sentence Ψ that is false in the Standard-M model. Then Ψ^T is also false under the Standard-M model (since Ψ and Ψ^T are equivalent in this model).

Hence Part-b of the definition of ξ –Conventionality implies α must prove $\neg \Psi^T$ (on account of Ψ^T 's Π_1^ξ format).

It is at this juncture that our proof-by-contradiction will reach its end. This is because if α can prove (51)'s statement and *also prove* the sentence $\neg \Psi^T$, then it certainly can combine these two facts to prove the non-existence of a proof of Ψ . The latter contradicts Part-a of the definition of ξ -Conventionality (because it shows α can verify its own consistency). \square

Remark E.2. We remind the reader that Footnote 14 pointed out that T 's translational mapping would lose its main functionality, if it did not require Ψ^T to have a Π_1^ξ format, similar to Ψ . In essence, Theorem E.1 is of interest because it shows that Theorem 6.12's evasion of the Second Incompleteness Theorem is close to being a maximal result. (It thus shows that (51)'s translational reflection principle does not generalize to conventional axiom systems.) This dichotomy may explain why self-justifying axiom systems, along with Theorem 6.12's particular invariant, are potentially useful results.

Appendix F: Epistemological Perspective and Speculations

It is desirable to include a short purely epistemological discussion within this mostly mathematical article so that the more subtle nature of our results cannot be misconstrued.

Part of the reason Self Justification can lend itself to easy misinterpretations is that the First Incompleteness Theorem demonstrates the impossibility of constructing an ideally optimal axiomatization of number theory. For any initial r.e. axiom system α and deduction method d , Gödel thus noted it is easy³⁶ to develop an extension of α that can prove strictly more theorems than α under d 's deduction method. Moreover, a large number of generalizations of the Second Incompleteness Theorem, starting with its 1939 Hilbert-Bernays version [16], are known to be robust results.

Such considerations naturally lead to questions about whether any r.e. axiom system can encompass the workings of the human mind. It may surprise some readers to learn that this author shares such skepticism. That is, we doubt *any single ISOLATED* self-justifying r.e. logic can *fully* approximate the complex workings of the human mind.

In this short appendix, let us instead view cogitation as *roughly* a process wondering though some universe \mathcal{U} , comprised of *both* consistent and inconsistent axiom systems, with a trial-and-error evolutionary method focusing its attention over time increasingly onto the members of this universe \mathcal{U} that are found to be consistent. It is straightforward³⁷ to define many universes \mathcal{U} and evolutionary processes that fall into

³⁶Let $\mathcal{U}(\alpha, d)$ the classic Gödel sentence that asserts: “*There is no proof of this sentence from α 's axiom system under d 's deduction method.*” Gödel [14] noted $\alpha + \mathcal{U}(\alpha, d)$ always proves more theorems than α .

³⁷It is trivial from a theoretical perspective to design a learning heuristic that will utilize all consistent

this genre. Our goal in this section will be to examine Section 5's "R-View" θ and its RE-Class(ξ).

Thus, θ will denote an R-View that consists of an arbitrary r.e. set of Π_1^ξ sentences. Also, RE-Class(ξ) will again denote the set of all θ which can be built under ξ 's language of L^ξ . (Section 5 had allowed both valid and invalid R-Views θ to appear in RE-Class(ξ) because no recursive decision procedure can identify all the Standard-M model's true Π_1^ξ sentences.)

The epistemological purpose of this notation was revealed in Section 6. For the cases where $k = 0$ or 1 , Section 6 defined $G_k^\xi(\theta)$ to be the axiom system:

$$G_k^\xi(\theta) = \theta \cup B^\xi \cup \text{SelfCons}^k\{[\theta \cup B^\xi], d\} \quad (52)$$

Also, Definition 6.2 indicated that the function G_k^ξ (which maps θ onto $G_k^\xi(\theta)$) would be called **Consistency Preserving** iff $G_k^\xi(\theta)$ is assured to be consistent whenever all the sentences in θ are true under the Standard-M model. Theorem 6.3 indicated, in this context, that G_1^ξ satisfies this property whenever ξ is EA-stable. Likewise, G_0^ξ is consistency preserving whenever ξ is one of A-stable, E-stable or 0-stable.

These results indicate a trial-and-error experimental process can, indeed, walk *in an unusually orderly manner* through an universe of self-reflecting candidate formalisms, when RE-Class(ξ) denotes \mathcal{U} 's universe and ξ satisfies any of the EA-stable, E-stable, A-stable or 0-stable conditions. This is because if θ designates a set of Π_1^ξ sentences holding true in the Standard-M model, then $G_k^\xi(\theta)$ will **automatically** satisfy both Parts (i) and (ii) of Section 1's definition of Self Justification, according to Theorem 6.3.

Such consistency preservation is surprising because it is simply inapplicable to the G_k^ξ functions for most pairs (ξ, k) . Theorem 6.3's first contribution is, thus, that it formalizes how G_k^ξ 's mapping function can represent a type of approximation for instinctive faith, under certain well-defined circumstances.

This notion of instinctive faith is, of course, less robust than a conventional proof. One obvious difficulty is that a 1-sentence proof, using an "*I am consistent*" axiom, is less convincing than a full-length proof from first principles. Also, if the initial formalism θ contains a false Π_1^ξ sentence then $B^\xi + \theta$ and $G_k^\xi(\theta)$ will be both inconsistent.

axiom systems from its available universe \mathcal{U} eventually, and it will spend only an infinitesimal fraction of its effort on inconsistent systems as time runs to infinity. (This because there exists only a countable number of distinct r.e. sets belonging to the universe \mathcal{U} .) Also, this learning process can presumably be made to employ some type of smart souped-up AI heuristics to enhance its efficiency, whose details will not concern us in this abbreviated 3-page appendix. What is central to the current discussion is that some type of formally *non-recursive* and presumably trial-and-error method must obviously be used by this learning process to find the consistent elements of \mathcal{U} , on account of Gödel's undecidability results.

Nevertheless for k equals 0 or 1, if θ is comprised of the true sentences in the Standard-M model, then Theorem 6.3 will assure that $G_k^\xi(\theta)$ is a consistent system that has an ability to use its “*I am consistent*” axiom sentence to formalize its own consistency. Moreover, the axiom system $G_k^\xi(\theta)$ is helpful because Gödel’s famous centennial paper implicitly raised the following bedeviling issue:

How is it that Human Beings manage to muster the physical drive to think (and prove theorems) when the many generalizations of Gödel’s Second Incompleteness Theorem demonstrate conventional logics lack knowledge of their own consistency?

While philosophical paradoxes and ironical dilemmas, similar to #, never yield perfect answers, the preceding discussion is helpful because it explores a certain syllogism whereby a logic can formalize at least some fragmented operational appreciation of its own consistency.

Moreover, Part-3 of Appendix D indicated that its four self-justifying configurations were close to being maximal results that cannot be much improved, on account of various barriers imposed by the Second Incompleteness Theorem. Thus, these particular positive results, combined with Theorems 5.9 6.3, 6.6, 6.10, 6.12, D.4, E.1, G.2, G.3 and Remarks 6.4 and 6.16, come close to formalizing the maximal variants of instinctive faith that a first-order logic can bolster.

The theme of the last two paragraphs is thus that our approximation of “*instinctive faith*” may be imperfect, but it is still a useful partial reply to #’s puzzling dilemma *in a context where* unambiguous full resolutions to # *are not permitted by* the Second Incompleteness Theorem. Furthermore, Equation (34)’s translational reflection principle, together with Theorem 6.12 and the Remarks 6.13 and 6.14, illustrate how the notion of an instinctive faith about the usefulness of Π_1^ξ theorems can be almost physically *hard-wired* into self-justifying formalisms.

A Yet Further Facet of this Unusual Epistemological Interpretation: Let the term *Epistemological Bundle Theory* refer to the underlying theory, advanced in this appendix, which speculates about a Thinking Agent walking through RE-Class(ξ)’s bundled universe of valid and invalid collections of Π_1^ξ sentences and then applying some heuristic to attempt to identify those $\theta \in \text{RE-Class}(\xi)$ whose sentences are true under the Standard-M model.

Such a theory has a second virtue, aside from addressing #’s paradoxical question about the nature of “*instinctive faith*”. It also clarifies the meaning of our main theorems and the related E-stability, A-stability, EA-stability and RE-Class(ξ) constructs.

This is because the Items $*$ and $**$ from the definitions of A-stability and E-stability in Section 5 formalize how a thinking agent T can view short proofs from a *technically inconsistent* axiom system of $B^\xi \cup \theta$ as containing pragmatically useful information *under the assumption* that the lengths of T 's proofs *are shorter* than the errors in θ 's Π_1^ξ styled-statements. The pleasing aspect about this observation, illustrated by Remark 5.2, is that those same invariants, $*$ and $**$, which tempt a thinking agent T to engage in a trial-and-error walk through RE-Class(ξ)'s bundled universe, also make viable Theorem 5.9's self-justifying formalisms.

Thus aside from addressing $\#$'s dilemma about the nature of instinctive faith, the meta-formalism in this appendix is useful in explaining the motivation behind the elaborate network of theorems, proofs and definitions that were introduced in this paper. In summary, EA-stable logics are thus interesting both in their own right (as a vehicle enabling a Thinking Being to partially tolerate its own errors), and because they are useful in explaining how a Thinking Being can possess a type of instinctive faith in its own consistency (via the reflection principles of Theorem 6.12 and of Remarks 6.13 and 6.14).

Appendix G: Improvements upon Theorems 6.6 and 6.10

Let us recall that Remark 6.11 indicated that there was a subtle trade-off between Theorems 6.6 and 6.10, where neither result was strictly better than the other. This section will introduce two hybrid methodologies, using Definition G.1's formalism, that improve upon Theorem 6.10 while retaining a large part of Theorem 6.6's nice features.

Definition G.1 Let ξ denote the generic configuration, whose base axiom system is again denoted as B^ξ , Φ denote any Π_1^ξ sentence that is true in the Standard-M model and j denote an index that represents some predicate Test_j^ξ lying in Definition 6.8's TestList^ξ sequence. Then a Π_1^ξ sentences Ψ will be said to be a **Braced** $^\xi(\Phi, j)$ expression when $B^\xi + \Phi$ can prove:

$$\{ \forall x \text{ Test}_j^\xi(\lceil \Psi \rceil, x) \} \longrightarrow \Psi \quad (53)$$

Theorem G.2 Let ξ again denote an arbitrary generic configuration $(L^\xi, \Delta_0^\xi, B^\xi, d, G)$, and let $(\mathcal{B}, \mathcal{D})$ again denote any second axiom system and deduction method whose Π_1^ξ theorems are true under the Standard-M model. Then for any integer j and for any Π_1^ξ sentence Φ that is true in the Standard-M model, the following invariants do hold:

- i If ξ is EA-stable then there will exist a self-justifying $\beta_j \supset B^\xi$ that can recognize its $\text{Level}(1^\xi)$ consistency, contains only a **finite number** of additional axioms beyond those appearing in B^ξ , and which can prove all of $(\mathcal{B}, \mathcal{D})$'s Π_1^ξ theorems that are **Braced** $^\xi(\Phi, j)$ expressions.

- ii) Likewise, if ξ is E -stable, A -stable or 0 -stable then a self-justifying $\beta_j \supset B^\xi$ will exist with the same properties except that it recognizes its own $\text{Level}(0^\xi)$ consistency.

Proof. To justify Theorem G.2, we must first define the axiom system β_j , whose existence is claimed by Items (i) and (ii). It will be defined to consist of the union of the initial base axiom system B^ξ with the following three added axiom-sentences.

- 1 The Π_1^ξ sentence Φ used by Definition G.1's $\text{Braced}^\xi(\Phi, j)$ formula.
- 2 A $\text{GlobSim}_B^D(\xi, j)$ sentence whose indexing integer j is defined by Definition G.1. This global simulation sentence is thus the statement:

$$\forall t \forall q \forall x \{ [\text{Prf}_B^D(t, q) \wedge \text{Check}^\xi(t)] \longrightarrow \text{Test}_j^\xi(t, x) \} \quad (54)$$

- 3 A Π_1^ξ sentence of the form $\text{SelfCons}^k\{ [\theta \cup B^\xi] , d \}$ where:
 - a θ is an R-view consisting of the two Π_1^ξ sentences defined by Items 1 and 2.
 - b B^ξ is ξ 's base axiom system, and
 - c k equals respectively 1 and 0 under formalisms (i) and (ii).

Thus, the system β_j uses identical definitions under formalisms (i) and (ii), except that its third sentence will use a different value for k . Our proof of Theorem G.2 will require first confirming the following fact:

Claim * The axiom system β_j (which consists of the union of B^ξ with the sentences 1-3) will have a capacity to prove every $\text{Braced}^\xi(\Phi, j)$ sentence Ψ that is a Π_1^ξ theorem of $(\mathcal{B}, \mathcal{D})$.

The proof of Claim * is quite simple. It will rest on the following three observations:

- a For each Π_1^ξ sentence Ψ , the system β_j must certainly have a capacity to prove (55)'s sentence (which states that Ψ 's Gödel number formally encodes a Π_1^ξ statement). This is because (55) is true in the Standard-M model and because Part 3 of Definition 3.4 indicated that the B^ξ sub-component of β_j has a capacity to prove every Δ_0^ξ sentence that is true.

$$\text{Check}^\xi([\Psi]) \quad (55)$$

- b Since Claim * specifies Ψ is a theorem of $(\mathcal{B}, \mathcal{D})$, there must certainly exist some integer N that is the Gödel number of its proof from $(\mathcal{B}, \mathcal{D})$. This implies that (56) must be a true Δ_0^ξ sentence under the Standard-M model. As was the case with Equation (55), this implies that it must be provable from B^ξ (because it is a valid Δ_0^ξ sentence).

$$\text{Prf}_B^D([\Psi] , N) \quad (56)$$

- c It is apparent that Equations (54), (55) and (56) imply the validity of (57). Moreover, Part 4 of Definition 3.4 indicated that the generic configuration ξ 's deduction method does satisfy Gödel's Completeness Theorem. This fact assures that β_j must be able to prove (57) because it contains (54) as an axiom and (55) and (56) as derived theorems ³⁸.

$$\forall x \quad \text{Test}_j^\xi(\lceil \Psi \rceil, x) \quad (57)$$

Claim * is a consequence of Observations a-c. This is because Φ is one of β_j 's defined axioms, and Definition G.1 indicated $B^\xi + \Phi$ was capable of proving (53)'s statement for every $\text{Braced}^\xi(\Phi, j)$ sentence Ψ . These facts corroborate Claim * because they imply that β_j must be able to verify Claim *'s sentence Ψ (because β_j can verify statements (53) and (57)).

The remainder of Theorem G.2's proof is analogous to Theorem 6.10's proof. This is because the prior paragraph established that β_j can prove every $\text{Braced}^\xi(\Phi, j)$ theorem of $(\mathcal{B}, \mathcal{D})$ (as was required by Claims i and ii). The only remaining task is to show that β_j is a self-justifying formalism that can recognize its $\text{Level}(1^\xi)$ and $\text{Level}(0^\xi)$ consistencies, as specified by Claims i and ii. This part of Theorem G.2's verification is identical to the methods used to prove Theorems 6.3 and 6.10. It will thus not be repeated here. \square

The last part of this appendix will require the following additional notation to formalize the main intended application of Theorem G.2's formalism.

1. $\text{Count}(\Psi)$ will denote the number of quantifiers appearing in the sentence Ψ (including both its bounded and unbounded quantifiers).
2. $\text{Size}^\xi(c)$ will denote the set of Π_1^ξ sentences Ψ where $\text{Count}(\Psi) \leq c$.

Our next theorem will be a specialized variant of Theorem G.2, using the $\text{Size}^\xi(c)$ construct. It will explain the intended application of this formalism:

Theorem G.3. *Let ξ denote any one of Appendix D's four sample generic configurations of ξ^* , ξ^{**} , ξ^- or ξ^R . Then for any $c > 0$, Theorem G.2's axiom systems of β_j can be arranged so that they can prove all of $(\mathcal{B}, \mathcal{D})$'s $\text{Size}^\xi(c)$ Π_1^ξ theorems while simultaneously also recognizing their:*

1. *Level(1) consistency for the cases when ξ is one of ξ^* , ξ^{**} or ξ^- .*
2. *Level(0) consistency when ξ is ξ^R .*

³⁸ Every deduction method d , satisfying Gödel's Completeness Theorem, will be automatically able to prove a theorem Z when it contains X, Y and $(X \wedge Y) \rightarrow Z$ as theorems, irregardless of whether or not it contains an explicit built-in modus ponens rule. Thus d can prove (57) because of its knowledge about (54)–(56)'s validity.

Proof Sketch: The intuition behind Theorem G.3's proof is quite easy to summarize. For arbitrary $c > 0$ and any of Appendix D's configurations of ξ^* , ξ^{**} , ξ^- and ξ^R , it is routine to construct an ordered pair (Φ, j) where every Π_1^ξ sentence of $\text{Size}^\xi(c)$ is a $\text{Braced}^\xi(\Phi, j)$ expression. Theorem G.3's first claim is, thus, a consequence of Part (i) of Theorem G.2 and the fact that each of ξ^* , ξ^{**} and ξ^- are EA-stable. Likewise, Theorem G.3's second claim follows from Part (ii) of Theorem G.2 and the fact that ξ^R is E-stable, \square

Remark G.4. The Theorems G.2 and G.3 are of interest because the set of Π_1^ξ sentences of $\text{Size}^\xi(c)$ is a natural class to examine. It is, thus, tempting to consider a system that recognizes its own formal consistency, uses only a finite number of axiom sentences beyond those in B^ξ , and which can prove all of $(\mathcal{B}, \mathcal{D})$'s Π_1^ξ theorems of $\text{Size}^\xi(c)$. Such a system replies to Remark 6.11's challenge by hybridizing the properties of Theorems 6.6 and 6.10, in a seemingly pragmatic manner.

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